# AdS string amplitudes from single-valuedness

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Based on work with:

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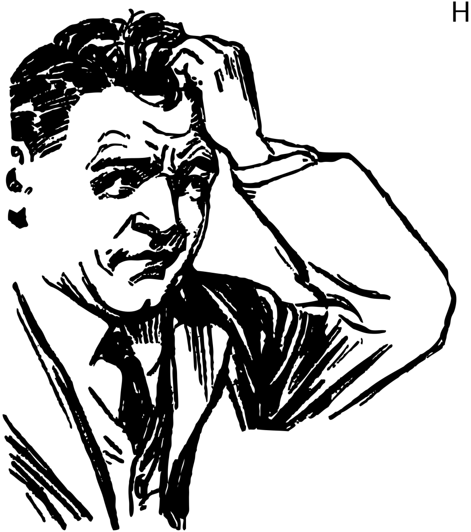


How to formulate string theory on curved spacetime?

At least for $AdS_5/CFT_4$?

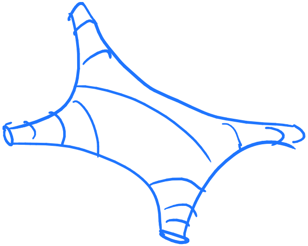
WWVD? – Fix the amplitude first!

**Outline:**

1. String amplitudes in flat space
2. Single-valuedness
3. AdS Virasoro-Shapiro amplitude
   4 gravitons, type IIB superstring in $AdS_5 \times S^5$ / $\mathcal{N} = 4$ SYM
4. AdS Veneziano amplitude
   4 gluons, orientifold of type IIB in $AdS_5 \times S^5$ / $\mathcal{N} = 2$ SCFT
5. High energy limit

# 1. String amplitudes in flat space

# String amplitudes depend on...



… the parameters of the theory:

- $g_s$ = string coupling $\ll 1$
  $\rightarrow$ consider tree level = genus 0
- $\sqrt{\alpha'}$ = string length

… the particles being scattered

- consider 4 gravitons (closed strings) or 4 gluons (open strings)
- momenta $p_i$ in terms of Mandelstams $S + T + U = 0$

$$S \sim \alpha'(p_1 + p_2)^2 \qquad T \sim \alpha'(p_1 + p_3)^2 \qquad U \sim \alpha'(p_1 + p_4)^2$$

- polarizations $\epsilon_i$

# Famous string amplitudes

4 gravitons in type IIB superstring: $\mathcal{A} = K_{\text{closed}}(\epsilon_i, p_i) A^{(0)}_{\text{closed}}(S, T)$

## Virasoro-Shapiro amplitude

$$A^{(0)}_{\text{closed}}(S, T) = -\frac{\Gamma(-S)\Gamma(-T)\Gamma(-U)}{\Gamma(S+1)\Gamma(T+1)\Gamma(U+1)}$$

4 gluons in type I superstring: $\mathcal{A} = K_{\text{open}}(\epsilon_i, p_i) \left( \text{Tr}(t^{i_1} t^{i_2} t^{i_3} t^{i_4}) A^{(0)}_{\text{open}}(S, T) + \text{permutations} \right)$

## Veneziano amplitude

$$A^{(0)}_{\text{open}}(S, T) = -\frac{\Gamma(-S)\Gamma(-T)}{\Gamma(1-S-T)}$$

# Partial wave expansion

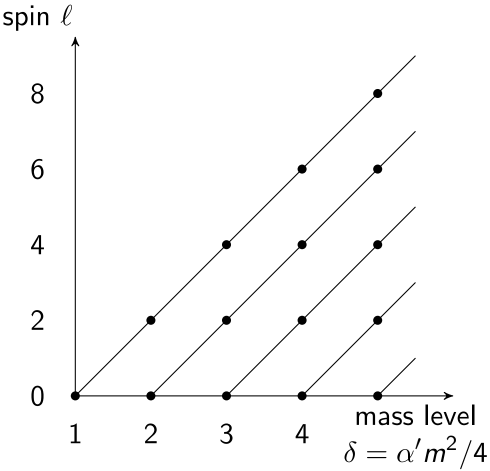
The exchanged massive string spectrum is extracted via the partial wave expansion

$$\lim_{T\to\delta} A^{(0)}(S,T) = \sum_{\ell} \frac{a_{\delta,\ell} P_\ell(\cos\theta)}{T-\delta}$$

It forms linear Regge trajectories.

$$A^{(0)}_{\text{closed}} = -\frac{\Gamma(-S)\,\Gamma(-T)\,\Gamma(-U)}{\Gamma(S+1)\Gamma(T+1)\Gamma(U+1)}$$

Spectrum for $A^{(0)}_{\text{closed}}(S,T)$:



# Low energy expansion

Low energy effective action (point particles with derivative interactions)
$\rightarrow$ Low energy expansion ($S \sim T \sim 0 \leftrightarrow$ short strings):

$$A^{(0)}_{\text{closed}}(S,T) = \frac{1}{STU} + 2\,\zeta(3) + \zeta(5)\,(S^2 + T^2 + U^2) + 2\,\zeta(3)^2\,STU + \dots$$

sugra $\quad R^4 \quad D^4R^4 \quad D^6R^4$

$$A^{(0)}_{\text{open}}(S,T) = -\frac{1}{ST} + \zeta(2) + \zeta(3)\,(S+T) + \zeta(4)\,(S^2 + \tfrac{1}{4}ST + T^2) + \dots$$

SYM $\quad F^4 \quad D^2F^4 \quad D^4F^4$

The LEE of closed string amplitudes contains only odd zeta-values!

This has a deep mathematical reason!
[Stieberger;2013],[Brown,Dupont;Schlotterer,Schnetz;Vanhove,Zerbini;2018]

# 2. Single-valuedness

# Zeta values and polylogarithms

Zeta values are related to polylogs:

$$\left.\begin{array}{lrcl} \text{zeta values:} & \zeta(n) & = & \displaystyle\sum_{k=1}^{\infty} \frac{1}{k^n} \\ \text{polylogarithms:} & \mathrm{Li}_n(z) & = & \displaystyle\sum_{k=1}^{\infty} \frac{z^k}{k^n} \end{array}\right\} \quad \zeta(n) = \mathrm{Li}_n(1)$$

Let's talk about polylogs.

# Multiple polylogarithms (MPLs)

**Definition $(|z_1 \dots z_r| = r = \text{weight})$** $z_i \in \{0, 1\}$

$$L_{z_1 \dots z_r}(z) = \int_{0 \le t_r \le \dots \le t_1 \le z} \frac{dt_1}{t_1 - z_1} \cdots \frac{dt_r}{t_r - z_r}$$

Properties:

- $\partial_z L_{z_i w}(z) = \frac{1}{z - z_i} L_w(z)$
- multi-valued
- holomorphic

Examples:

- $L_{1^p}(z) = \frac{1}{p!} \log^p(1 - z)$
- $L_{0^p 1}(z) = -\mathrm{Li}_{p+1}(z)$

# Single-valued multiple polylogarithms (SVMPLs)

**SVMPLs** [Brown;2004]

$$\mathcal{L}_w(z) = \sum_{|w_1|+|w_2|=|w|} c_{w_1 w_2} L_{w_1}(z) L_{w_2}(\bar{z})$$

Properties:

- $\partial_z \mathcal{L}_{z_i w}(z) = \frac{1}{z - z_i} \mathcal{L}_w(z)$
- single-valued
- non-holomorphic

Examples:

- $\mathcal{L}_{1^p}(z) = \frac{1}{p!} \log^p |1 - z|^2$
- $\mathcal{L}_{01}(z) = \mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z}) - \log(1 - \bar{z}) \log |z|^2$

# Single-valued zeta values

**Single-valued zeta values** [Brown;2013]

$$\zeta_{\mathrm{sv}}(w) \equiv \mathcal{L}_w(1)$$

Subset of the usual multiple zeta values.

In particular

$$\zeta_{\mathrm{sv}}(2n+1) = 2\zeta(2n+1) \qquad \zeta_{\mathrm{sv}}(2n) = 0$$

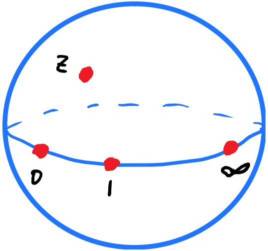
Odd zeta values are single-valued, even ones are not!

Example for multiple zetas (nested sums):

$$\zeta_{\mathrm{sv}}(3,5,3) = 2\zeta(3,5,3) - 2\zeta(3)\zeta(3,5) - 10\zeta(3)^2\zeta(5)$$

# Single-valuedness from worldsheet integrals

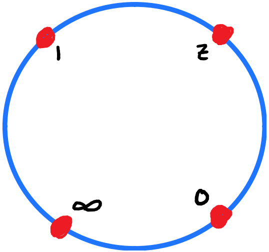
**Closed strings**



$$A^{(0)}_{\text{closed}} = \frac{1}{U^2} \int dz^2 \; |z|^{-2S-2} |1-z|^{-2T-2}$$
$$= \frac{1}{STU} + 2\,\zeta(3) + \dots$$

Single-valued!

**Open strings**



$$A^{(0)}_{\text{open}} = -\frac{1}{U} \int_0^1 dz \; z^{-S-1} (1-z)^{-T-1}$$
$$= -\frac{1}{ST} + \zeta(2) + \dots$$

Not single-valued!

Certain (2-dimensional) integrals preserve single-valuedness!

[Brown,Dupont;Schlotterer,Schnetz;Vanhove,Zerbini;2018]

# Strings in (weakly) curved background

Consider curvature corrections to amplitudes: ($R$ = curvature scale)

$$A(S,T) = A^{(0)}(S,T) + \frac{\alpha'}{R^2} A^{(1)}(S,T) + \dots$$

Toy non-linear sigma model:

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{g}\, g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \, G_{\mu\nu}(X) \longleftarrow$$

$$= S_{\text{flat}} + \frac{1}{R^2} \lim_{q\to 0} \frac{\partial^2}{\partial q^\mu \partial q^\nu} V^{\mu\nu}_{\text{graviton}}(q) + \dots$$

Curved metric expanded around flat space:

$$G_{\mu\nu}(X) = \eta_{\mu\nu} + \frac{h_{\mu\nu}}{R^2} + \cdots$$

$$h_{\mu\nu} \sim X_\mu X_\nu \sim \lim_{q\to 0} \frac{\partial^2}{\partial q^\mu \partial q^\nu} e^{iq\cdot X}$$

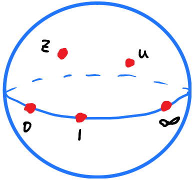
curvature corrections $\sim$ extra soft gravitons

$$A^{(1)}(S,T) \sim \lim_{q\to 0} \frac{\partial^2}{\partial q^\mu \partial q^\nu} \left\langle V_1 V_2 V_3 V_4 V^{\mu\nu}_{\text{graviton}}(q) \right\rangle_{\text{flat}}$$

# Worldsheet integrals for curvature corrections

String amplitudes with an extra soft graviton:

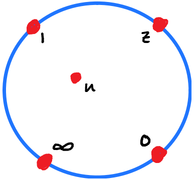
$$A^{(1)}_{\text{closed}} \sim$$



$$\sim \int dz^2 \, |z|^{-2S-2} |1-z|^{-2T-2} \underbrace{\int d^2u \frac{|u|^{2p_1\cdot q}|1-u|^{2p_3\cdot q}|z-u|^{2p_2\cdot q}}{|u|^2|1-u|^2}}_{}$$

In a small $q$ expansion: SVMPLs($z$)

$$A^{(1)}_{\text{open}} \sim$$



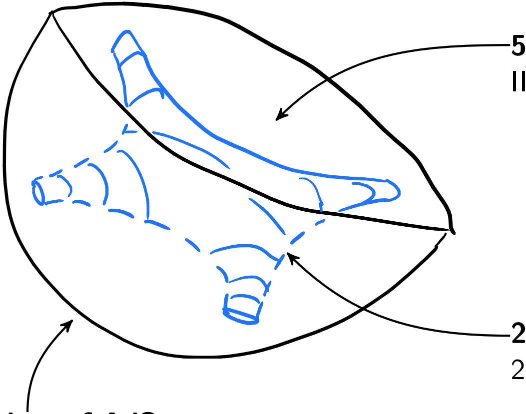
$$\sim \int_0^1 dz \, z^{-S-1}(1-z)^{-T-1} \overbrace{\int d^2u \frac{|u|^{2p_1\cdot q}|1-u|^{2p_3\cdot q}|z-u|^{2p_2\cdot q}}{|u|^2|1-u|^2}}^{}$$

The curvature corrections $A^{(k)}$ should be world-sheet integrals over SVMPLs!

Next we will use AdS/CFT to make this precise!

# 3. AdS Virasoro-Shapiro amplitude

# AdS$_5$ / CFT$_4$



**5d bulk of AdS:**
IIb string theory on $AdS_5 \times S^5$

- string coupling $g_s$
- string length $\sqrt{\alpha'}$
- AdS radius $R$

**2d string worldsheet:**
2d CFT???

**4d boundary of AdS:**
$\mathcal{N} = 4$ super Yang Mills theory

- $SU(N)$ gauge group
- coupling $\sqrt{\lambda} = \dfrac{R^2}{\alpha'}$

Weakly coupled strings:

$$g_s \ll 1 \quad \Leftrightarrow \quad N \gg 1$$

# The AdS Virasoro-Shapiro amplitude

**CFT stress-tensor correlator**

$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle$

↔

integral transform

**AdS graviton amplitude**

$A_{\text{closed}}(S,T)$

Small curvature expansion:

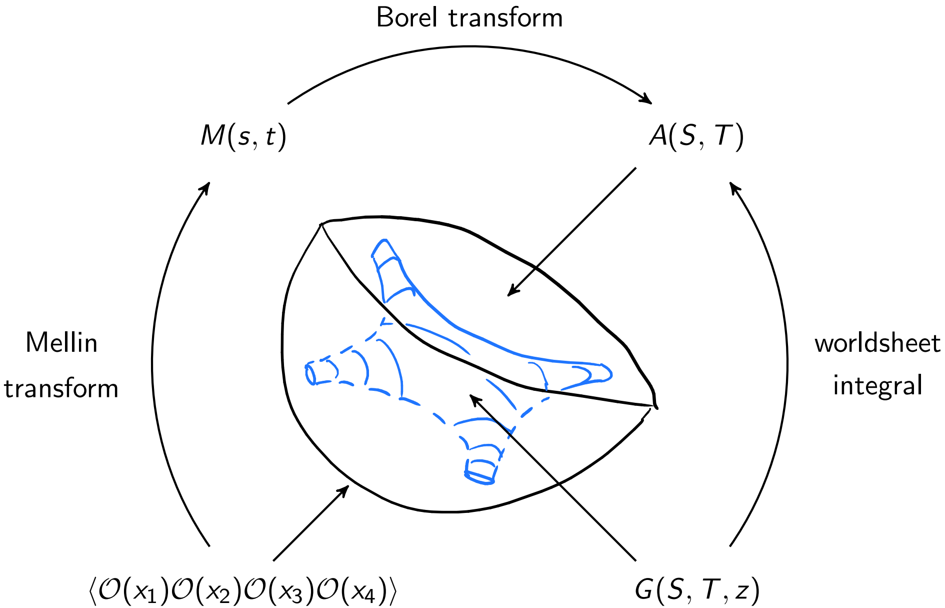
$$A_{\text{closed}}(S,T) = A^{(0)}_{\text{closed}}(S,T) + \frac{\alpha'}{R^2} A^{(1)}_{\text{closed}}(S,T) + \left(\frac{\alpha'}{R^2}\right)^2 A^{(2)}_{\text{closed}}(S,T) + \ldots$$

$$A^{(0)}_{\text{closed}}(S,T) = -\frac{\Gamma(-S)\Gamma(-T)\Gamma(-U)}{\Gamma(S+1)\Gamma(T+1)\Gamma(U+1)}$$

$R =$ AdS radius

$\frac{\alpha'}{R^2} = \frac{1}{\sqrt{\lambda}}$ ← t'Hooft coupling

# Integral transforms



Borel transform

$M(s,t)$

$A(S,T)$

Mellin transform

worldsheet integral

$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle$

$G(S,T,z)$

# The Mellin transform

Cross-ratios: $U = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$ $V = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$

## Mellin transform

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle \propto \int_{-i\infty}^{i\infty} dsdt\, U^s V^t\, \Gamma(s,t)\, M(s,t)$$

$$\begin{array}{ccc} \langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle & & M(s,t) \\ \text{powers of } U, V & \leftrightarrow & \text{poles in } s, t \end{array}$$

Mellin amplitudes share many properties of scattering amplitudes.

# Operator product expansion

We can expand $\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle$ into conformal blocks using:

### Operator product expansion (OPE)

$$\mathcal{O}(x)\mathcal{O}(0) = \sum_{\mathcal{O}_{\Delta,\ell} \text{ primaries}} C_{\Delta,\ell}\; c_{\Delta,\ell}(x,\partial_y)\mathcal{O}_{\Delta,\ell}(y)\Big|_{y=0}$$

### OPE data

- $\Delta =$ dimension
- $\ell =$ spin
- $C_{\Delta,\ell} =$ OPE coefficients

$M(s,t)$ has only simple poles, given by [Mack;2009], [Penedones,Silva,Zhiboedov;2019]

### Poles and residues of $M(s,t)$

$$M(s,t) \sim \frac{C^2_{\Delta,\ell} Q_{\Delta,\ell,m}(t)}{s-(\Delta-\ell+2m)}$$

# Massive string operators

String masses in flat space:

$$m^2 = \frac{4\delta}{\alpha'}\,, \quad \delta = 1, 2, 3, \ldots$$

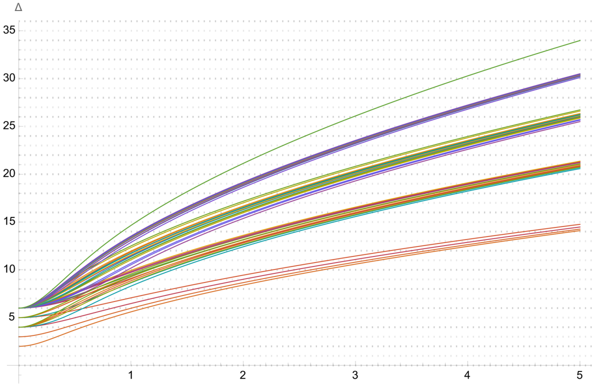
AdS dictionary ($\sqrt{\lambda} = R^2/\alpha' \gg 1$):

$$\Delta(\Delta - d) = R^2 m^2 + O(\lambda^0) = R^2 \frac{4\delta}{\alpha'} + O(\lambda^0)$$

Expanded OPE data:

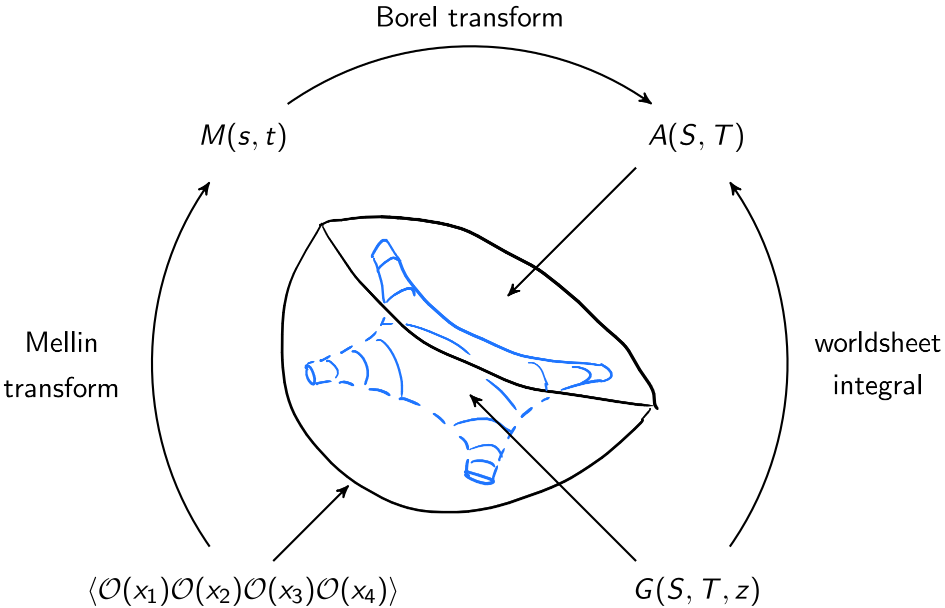
$$\begin{array}{rcccccc} & & A^{(0)}_{\text{closed}} & & A^{(1)}_{\text{closed}} & & A^{(2)}_{\text{closed}} \\ \Delta_{\delta,\ell} = & & 2\sqrt{\delta}\lambda^{\frac{1}{4}} & + & \lambda^{-\frac{1}{4}}\Delta^{(1)}_{\delta,\ell} & + & \lambda^{-\frac{3}{4}}\Delta^{(2)}_{\delta,\ell} \\ C^2_{\delta,\ell} = & & C^{2(0)}_{\delta,\ell} & + & \lambda^{-\frac{1}{2}}C^{2(1)}_{\delta,\ell} & + & \lambda^{-1}C^{2(2)}_{\delta,\ell} \end{array}$$

Integrability:
unprotected operators (Konishi etc.)
from weak to strong coupling:



[Gromov,Hegedus,Julius,Sokolova;2023]

# Integral transforms (again)



Borel transform

$M(s, t)$

$A(S, T)$

Mellin transform

worldsheet integral

$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle$

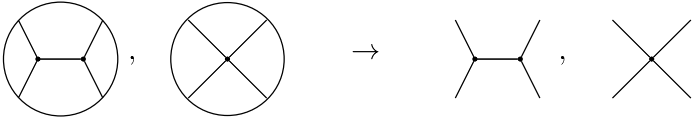
$G(S, T, z)$

# The Borel transform

## Borel transform / flat space limit

$$A(S,T) = \lambda^{\frac{3}{2}} \int_{-i\infty}^{i\infty} \frac{d\alpha}{2\pi i}\, e^{\alpha} \alpha^{-6} M\left(\frac{2\sqrt{\lambda}S}{\alpha}, \frac{2\sqrt{\lambda}T}{\alpha}\right)$$

1. Maps Witten diagrams to Feynman diagrams for $R \to \infty$ [Penedones;2010]



2. Borel summation of the low energy expansion:

$$M(s,t) = \sum_{p,q} \frac{\Gamma(6+p+q)}{\lambda^{\frac{3}{2}}} \left(\frac{s}{2\sqrt{\lambda}}\right)^p \left(\frac{t}{2\sqrt{\lambda}}\right)^q \alpha_{p,q} \quad \Rightarrow \quad A(S,T) = \sum_{p,q} S^p T^q \alpha_{p,q}$$

# Pole structure of the AdS amplitude

Borel transform of the OPE leads to:

### Pole structure of the AdS amplitude

$$A^{(k)}(S,T) = \frac{R^{(k)}_{3k+1}(T, \text{OPE data})}{(S-\delta)^{3k+1}} + \ldots + \frac{R^{(k)}_{1}(T, \text{OPE data})}{S-\delta} + O((S-\delta)^0)$$

The numerator functions are known explicitly.

# Worldsheet ansatz

## Ansatz

$$A^{(k)}_{\text{closed}}(S,T) = B^{(k)}(S,T) + B^{(k)}(U,T) + B^{(k)}(S,U)$$

$$B^{(k)}(S,T) = \frac{1}{(S+T)^2}\int d^2z|z|^{-2S-2}|1-z|^{-2T-2}G^{(k)}_{\text{closed}}(S,T,z)$$

Assumed properties of $G^{(k)}_{\text{closed}}(S,T,z)$:

- transcendental weight $3k$ (SVMPLs(z), SVMZVs)
- homogeneous degree $2k$ polynomial in $S$, $T$
- crossing symmetry: $G^{(k)}_{\text{closed}}(S,T,z) = G^{(k)}_{\text{closed}}(T,S,1-z)$

# Solving the constraints

Matching

### Worldsheet ansatz

$$\begin{aligned}G^{(1)}_{\text{closed}}(S,T,z) = (S+T)^2\left(c_1\mathcal{L}^{+}_{000}(z)+c_2\mathcal{L}^{+}_{001}(z)+c_3\mathcal{L}^{+}_{010}(z)+c_4\zeta(3)\right)\\ +ST\left(c_5\mathcal{L}^{+}_{000}(z)+c_6\mathcal{L}^{+}_{001}(z)+c_7\mathcal{L}^{+}_{010}(z)+c_8\zeta(3)\right)\\ +(S^2-T^2)\left(c_9\mathcal{L}^{-}_{000}(z)+c_{10}\mathcal{L}^{-}_{001}(z)+c_{11}\mathcal{L}^{-}_{010}(z)\right)\end{aligned}$$

and $\mathcal{L}^{\pm}_{w}(z)=\mathcal{L}_w(z)\pm\mathcal{L}_w(1-z)$

### OPE pole structure

$$A^{(1)}_{\text{closed}}(S,T)=\frac{R^{(1)}_4(T,\text{OPE data})}{(S-\delta)^4}+\ldots+\frac{R^{(1)}_1(T,\text{OPE data})}{S-\delta}+O((S-\delta)^0)$$

fixes all unknowns in both expressions!

# Worldsheet solution

First correction: ansatz has 11 rational parameters

## Solution

$$G^{(1)}_{\text{closed}}(S,T,z) = (S+T)^2\left(-\frac{1}{6}\mathcal{L}^{+}_{000}(z) - \frac{1}{4}\mathcal{L}^{+}_{010}(z) + 2\zeta(3)\right) + (S^2-T^2)\left(-\frac{1}{6}\mathcal{L}^{-}_{000}(z) + \frac{1}{3}\mathcal{L}^{-}_{001}(z) + \frac{1}{6}\mathcal{L}^{-}_{010}(z)\right)$$

Second correction: ansatz has 115 rational parameters

## Solution

$$G^{(2)}_{\text{closed}}(S,T,z) = \frac{1}{18}(S+T)^2(ST-S^2-T^2)\mathcal{L}^{+}_{000000}(z) + \text{ 44 more terms}$$

# Checks

Success! But we made assumptions...

There are direct connections to many other results:

| Quantity | Compare with |
|---|---|
| Wilson coefficients | supersymmetric localization |
| Conformal dimensions | integrability |
| OPE coefficients | numerical conformal bootstrap |
| High energy limit | classical string scattering in AdS |

Let's compare!

# Check 1: Low energy expansion

Relates to low energy effective action (SUGRA + derivative interactions)

$$A(S,T) = \mathrm{SUGRA} + \sum_{a,b,k=0}^{\infty} \frac{\sigma_2^a \sigma_3^b}{\sqrt{\lambda}^k} \alpha_{a,b}^{(k)} , \qquad \sigma_2 = S^2 + T^2 + U^2, \sigma_3 = STU$$

$$= \mathrm{SUGRA} + \underset{R^4}{\alpha_{0,0}^{(0)}} + \underset{D^2R^4}{\frac{\alpha_{0,0}^{(1)}}{\sqrt{\lambda}}} + \underbrace{\sigma_2\, \alpha_{1,0}^{(0)} + \frac{\alpha_{0,0}^{(2)}}{\lambda}}_{D^4R^4} + \underbrace{\sigma_3\, \alpha_{0,1}^{(0)} + \frac{\sigma_2\, \alpha_{1,0}^{(1)}}{\sqrt{\lambda}} + \frac{\alpha_{0,0}^{(3)}}{\sqrt{\lambda}^3}}_{D^6R^4} + \ldots$$

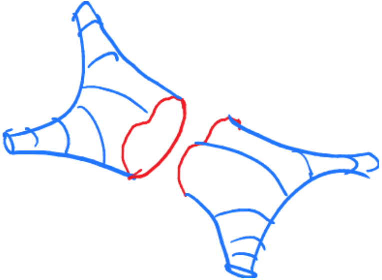
$\alpha_{a,b}^{(0)}$ = flat space, we fix all $\alpha_{a,b}^{(1)}$ and $\alpha_{a,b}^{(2)}$ , in particular:

$$\alpha_{0,0}^{(1)} = 0 , \quad \alpha_{1,0}^{(1)} = -\frac{22}{3}\zeta(3)^2 , \quad \alpha_{0,0}^{(2)} = \frac{49}{4}\zeta(5) , \quad \alpha_{1,0}^{(2)} = \frac{4091}{16}\zeta(7)$$

✓

Agrees with localisation result! Altogether we fully fix $D^8R^4$ and $D^{10}R^4$.
[Binder,Chester,Pufu,Wang;2019],[Chester,Pufu;2020],[Alday,TH,Silva;2022]

# Check 2: OPE data



We extract the OPE data:

$$\begin{array}{rcccccccl} & & A^{(0)}\text{ data} & & A^{(1)}\text{ data} & & A^{(2)}\text{ data} & & \\ \Delta_{\delta,\ell} & = & 2\sqrt{\delta}\lambda^{\frac{1}{4}} & + & \lambda^{-\frac{1}{4}}\Delta^{(1)}_{\delta,\ell} & + & \lambda^{-\frac{3}{4}}\Delta^{(2)}_{\delta,\ell} & + & \dots \\ C^2_{\delta,\ell} & = & C^{2(0)}_{\delta,\ell} & + & \lambda^{-\frac{1}{2}}C^{2(1)}_{\delta,\ell} & + & \lambda^{-1}C^{2(2)}_{\delta,\ell} & + & \dots \end{array}$$

Leading Regge trajectory ($\delta = 1$ is Konishi):

$$\Delta = 2\sqrt{\delta}\lambda^{\frac{1}{4}}\left(1 + \left(\frac{3\delta}{4} + \frac{1}{2\delta} - \frac{1}{4}\right)\frac{1}{\sqrt{\lambda}} - \left(\frac{21\delta^2}{32} + \frac{1}{8\delta^2} - \frac{(3-12\zeta(3))\delta}{8} - \frac{1}{8\delta} - \frac{17}{32}\right)\frac{1}{\lambda} + \dots\right)$$

Agrees with integrability result! ✓

[Gromov,Serban,Shenderovich,Volin;2011],[Basso;2011],[Gromov,Valatka;2011]

Konishi OPE coefficient agrees with bootstrap! [Caron-Huot,Coronado,Trinh,Zahraee;2024]

# 4. AdS Veneziano amplitude

# Open strings in AdS

Type IIB (on $AdS_5 \times S^5$) has only closed strings.

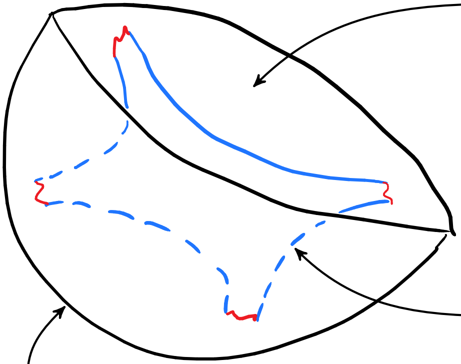
Q: What is the simplest $\mathrm{AdS}_5/\mathrm{CFT}_4$ with weakly coupled open strings?

A: An orientifold of type IIB! [Sen;1996],[Banks,Douglas,Seiberg;1996]

$= N$ $D3$-branes near $D_4$-type $F$-theory singularity

$=$ type IIB with $N$ $D3$'s, 4 $D7$'s, 1 $O7$

# $\mathrm{AdS}_5$ / $\mathrm{CFT}_4$ with open strings



**5d bulk of AdS:**
type IIB gluons on $AdS_5 \times S^3$

$G = SO(8)$ gauge group

parameters: $g_s \ll 1$, $R$, $\alpha'$

dictionary: $\dfrac{R^4}{\alpha'^2} = \lambda$

**2d string worldsheet:**
2d CFT???

**4d boundary of AdS:**
$\mathcal{N} = 2$ $USp(2N)$ gauge theory

$G = SO(8)$ flavour group

parameters: $N \gg 1$, $\lambda$

Other possibilities: orbifolds of this
$G = U(4)$ or $G = SO(4) \times SO(4)$
[Ennes,Lozano,Naculich,Schnitzer;2000]

# The AdS Veneziano amplitude

**CFT flavour multiplet correlator**

$\langle \mathcal{O}^{I_1}(x_1)\mathcal{O}^{I_2}(x_2)\mathcal{O}^{I_3}(x_3)\mathcal{O}^{I_4}(x_4)\rangle$

$\longleftrightarrow$ integral transform

**AdS gluon amplitude**

$A^{I_1 I_2 I_3 I_4}_{\text{open}}(S,T)$

Color ordered amplitude:

$$A^{I_1 I_2 I_3 I_4}_{\text{open}}(S,T) = \text{Tr}(t^{I_1} t^{I_2} t^{I_3} t^{I_4})\, A_{\text{open}}(S,T) + \text{permutations}$$

Small curvature expansion:

$$A_{\text{open}}(S,T) = A^{(0)}_{\text{open}}(S,T) + \frac{\alpha'}{R^2} A^{(1)}_{\text{open}}(S,T) + \left(\frac{\alpha'}{R^2}\right)^2 A^{(2)}_{\text{open}}(S,T) + \ldots$$

World-sheet integral:

$$A^{(k)}_{\text{open}}(S,T) = \frac{1}{S+T}\int_0^1 dz\; z^{-S-1}(1-z)^{-T-1} G^{(k)}_{\text{open}}(S,T,z)$$

# Worldsheet ansatz for AdS Veneziano

Ansatz:

$$G^{(k)}_{\text{open}}(S, T, z) = \frac{1}{(S+T)^k} \sum_{n=0}^{3k} \sum_{j} P_{n,j}(S, T)\;\; T_{n,j}(z)$$

homogeneous degree $n$ polynomials　　weight $n$ (SV)MPLs

| weight | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| # of MPLs($z$) | 1 | 2 | 5 | 11 | 23 | 48 | 98 |
| # of SVMPLs($\bar{z} = z$) | 1 | 2 | 3 | 7 | 11 | 22 | 39 |

$G^{(1)}$: pole structure fixes MPL ansatz → result also matches SVMPL ansatz

$G^{(2)}$: pole structure fixes SVMPL ansatz up to 1 coefficient

# Worldsheet solution for AdS Veneziano

First correction: mv/sv ansatz has 33/22 rational parameters

### Solution

$$
\begin{aligned}
G^{(1)}_{\text{open}}(S,T,z) = {} & \frac{S^2+T^2}{4}\left(\mathcal{L}^+_{000}(z)-\mathcal{L}^+_{001}(z)\right) - \frac{(S+T)^2}{4}\left(\mathcal{L}^+_{010}(z)-4\zeta(3)\right) \\
& - \frac{3S^2+8ST+3T^2}{4(S+T)}\mathcal{L}^+_{00}(z) + \frac{5S^2+12ST+5T^2}{4(S+T)}\mathcal{L}^+_{01}(z) + \frac{3}{4}\mathcal{L}^+_{0}(z) + \frac{3}{S+T} \\
& + \frac{S-T}{4}\left((S+T)\left(\mathcal{L}^-_{000}(z)-\mathcal{L}^-_{001}(z)-\mathcal{L}^-_{010}(z)\right) - 3\mathcal{L}^-_{00}(z) - \frac{5\mathcal{L}^-_{0}(z)}{S+T}\right)
\end{aligned}
$$

Second correction: sv ansatz has 254 rational parameters

### Solution

$$G^{(2)}_{\text{open}}(S,T,z) = \ldots$$

# Low energy expansion of AdS Veneziano

Relates to low energy effective action (SYM + derivative interactions)

$$A(S,T) = -\frac{1}{ST} + \sum_{a,b,k=0}^{\infty} \frac{\sigma_1^a \sigma_2^b}{\sqrt{\lambda}^k} \alpha_{a,b}^{(k)}\,, \qquad \sigma_1 = -U,\ \sigma_2 = -ST$$

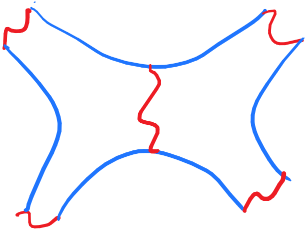
$$= \underset{\text{SYM}}{-\frac{1}{ST}} + \underset{F^4}{\alpha_{0,0}^{(0)}} + \underbrace{\sigma_1\, \alpha_{1,0}^{(0)} + \frac{\alpha_{0,0}^{(1)}}{\sqrt{\lambda}}}_{D^2F^4} + \underbrace{\sigma_1^2\, \alpha_{2,0}^{(0)} + \sigma_2\, \alpha_{0,1}^{(0)} + \frac{\sigma_1\, \alpha_{1,0}^{(1)}}{\sqrt{\lambda}} + \frac{\alpha_{0,0}^{(2)}}{\lambda}}_{D^4F^4} + \dots$$

$\alpha_{a,b}^{(0)}$ = flat space, we fix all $\alpha_{a,b}^{(1)}$ and $\alpha_{a,b}^{(2)}$ and the full $D^6F^4$ term.

Localization provides 1 constraint for each interaction term
$G = SO(8)$: [Behan,Chester,Ferrero;2023], $G = U(4)$: [Billo,Frau,Lerda,Pini,Vallarino;2024]

$\alpha_{0,0}^{(1)} = 0$ agrees, $\alpha_{0,0}^{(2)} = 48\zeta(2)^2$ fixes final # in $G_{\text{open}}^{(2)}(S,T,z)$ ✓

# OPE data from AdS Veneziano



We extract the OPE data:

$$\begin{array}{cccccccc} & & A^{(0)}\text{ data} & & A^{(1)}\text{ data} & & A^{(2)}\text{ data} & \\ \Delta_{\delta,\ell} & = & \sqrt{\delta}\lambda^{\frac{1}{4}} & + & \lambda^{-\frac{1}{4}}\Delta^{(1)}_{\delta,\ell} & + & \lambda^{-\frac{3}{4}}\Delta^{(2)}_{\delta,\ell} & +\dots \\ C^2_{\delta,\ell} & = & C^{2(0)}_{\delta,\ell} & + & \lambda^{-\frac{1}{2}}C^{2(1)}_{\delta,\ell} & + & \lambda^{-1}C^{2(2)}_{\delta,\ell} & +\dots \end{array}$$

Leading Regge trajectory:

$$\Delta = \sqrt{\delta}\lambda^{\frac{1}{4}}\left[\underbrace{1}_{0} + \left(\underbrace{\frac{3\delta}{4}+\frac{1}{2\delta}}_{0}\underbrace{-\frac{3}{4}}_{1}\right)\frac{1}{\sqrt{\lambda}} - \left(\underbrace{\frac{21\delta^2}{32}+\frac{1}{8\delta^2}}_{0}\underbrace{+\frac{(3+14\zeta(3))\delta}{4}-\frac{3}{8\delta}}_{1}\underbrace{-\frac{41}{32}}_{2}\right)\frac{1}{\lambda}+\dots\right]$$

0: matches classical solution for glued folded open string! ✓

$$\left.\begin{array}{l}\text{1: 1-loop fluctuation}\\ \text{2: 2-loop fluctuation}\end{array}\right\}\text{ open problem for semi-classics / integrability!}$$

# 5. High energy limit

# Why the high energy limit?

What is the next step towards the worldsheet theory?

Flat space [Gross,Mende;1987]:

classical solution (bosonic) of the worldsheet theory → high energy limit ($S, T \to \infty$) of string amplitudes

An independent way to compute $\lim_{S,T\to\infty} A(S,T)$, agnostic to many details!

# High energy limit via saddle point

The high energy limit of $A^{(0)}(S,T)$ is given by the saddle point $z=\bar{z}=\dfrac{S}{S+T}$

$$\lim_{S,T\to\infty}\int d^2z\,|z|^{-2S}|1-z|^{-2T}\sim e^{-2S\log|\frac{S}{S+T}|-2T\log|\frac{T}{S+T}|}$$

In AdS the limit can be computed in the same way.

Goal: Compute this exponent from the string action.

# Classical solution in flat space

Gross and Mende computed the high energy limit by minimizing the action

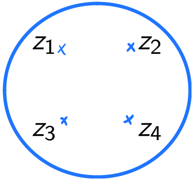
$$\mathcal{S}(X^\mu) = \int d^2\zeta \left( \partial X^\mu(\zeta) \bar{\partial} X_\mu(\zeta) - i \sum_{j=1}^{4} p_j \cdot X(\zeta)\, \delta^{(2)}(\zeta - z_j) \right)$$

EOM: $\partial\bar{\partial} X^\mu = -\frac{i}{2} \sum_j p_j^\mu \delta^{(2)}(\zeta - z_j)$ Virasoro: $\partial X \cdot \partial X = \bar{\partial} X \cdot \bar{\partial} X = 0$

Solution: $X^\mu_{\text{clas}} = -i \sum_j p_j^\mu \log|\zeta - z_j|$

This classical solution gives the correct high energy exponent:

$$\lim_{S,T\to\infty} A^{\text{flat}}(S,T) \sim e^{-\mathcal{S}(X^\mu_{\text{clas}})}\Big|_{z=\frac{S}{S+T}} = e^{-2S\log|\frac{S}{S+T}| - 2T\log|\frac{T}{S+T}|}$$



# The AdS path integral

The action for AdS:

$$\mathcal{S}(X,\Lambda) = \int d^2\zeta \left( \partial X^M \bar{\partial} X_M + \Lambda(X^M X_M + R^2) - i \sum_{j=1}^{4} P_j^M X_M \delta^{(2)}(\zeta - z_j) \right)$$

$AdS_d$ is embedded in $\mathbb{R}^{2,d-1} \ni X^M$

$$-R^2 = X^M X_M = -X^0 X^0 + X^\mu X_\mu$$

Eliminate $X^0$ and expand $X^\mu$ around flat space:

$$X^\mu = X_0^\mu + \frac{1}{R^2} X_1^\mu + \ldots \qquad\qquad X_0^\mu = -i \sum_j p_j^\mu \log \left| 1 - \frac{\zeta}{z_j} \right|$$

Equation of motion for $X_1^\mu$:

$$\partial\bar{\partial} X_1^\mu = \partial X_0 \cdot \bar{\partial} X_0 \, X_0^\mu = \frac{i}{4} \sum_{i,j,k} \frac{p_i \cdot p_j}{(\zeta - z_i)(\bar{\zeta} - z_j)} p_k^\mu \log \left| 1 - \frac{\zeta}{z_k} \right|$$

# Classical solution in AdS

Equation of motion for $X_1^\mu$:

$$\partial\bar{\partial}X_1^\mu = \partial X_0 \cdot \bar{\partial}X_0\, X_0^\mu = \frac{i}{8}\sum_{i,j,k}\frac{p_i \cdot p_j}{(\zeta - z_i)(\bar{\zeta} - z_j)} p_k^\mu \mathcal{L}_{z_k}(\zeta)$$

We can “integrate” this using

$$\int d\zeta \frac{\mathcal{L}_w(\zeta)}{\zeta - z_i} \to \mathcal{L}_{z_i w}(\zeta)\,, \qquad \int d\bar{\zeta} \frac{\mathcal{L}_w(\zeta)}{\bar{\zeta} - z_j} \to \mathcal{L}_{w z_j}(\zeta) + \cdots$$

Result:

$$X^\mu_{1,\text{clas}} = \frac{i}{8}\sum_{i,j,k=1}^{4} p_i \cdot p_j\; p_k^\mu \left(\mathcal{L}_{z_i z_k z_j}(\zeta) + \mathcal{L}_{z_k}(z_j)\mathcal{L}_{z_i z_j}(\zeta) - \mathcal{L}_{z_j}(z_k)\mathcal{L}_{z_i z_k}(\zeta)\right)$$

More generally:

$$X^\mu_{\text{clas}} = \mathcal{L}_{|w|=1}(\zeta) + \frac{1}{R^2}\mathcal{L}_{|w|=3}(\zeta) + \frac{1}{R^4}\mathcal{L}_{|w|=5}(\zeta) + \ldots$$

# Comparison with AdS Virasoro-Shapiro amplitude

$$e^{-\mathcal{S}(X^{\mu}_{\text{clas}})}\Big|_{z=\frac{S}{S+T}} = \exp\left(-SF_1\left(\tfrac{S}{T}\right) - \frac{S^2}{R^2}F_3\left(\tfrac{S}{T}\right) - \frac{S^3}{R^4}F_5\left(\tfrac{S}{T}\right) - O\left(\frac{S^4}{R^6}\right)\right)$$

In the limit $S, T, R \to \infty$ with $S/T$ and $S/R$ fixed, $F_5$ and further terms vanish!

We successfully compare with AdS Virasoro-Shapiro at the saddle point:

$$e^{-\frac{S^2}{R^2}F_3\left(\frac{S}{T}\right)} = 1 + \frac{1}{R^2}G^{(1)}_{\text{closed}}(z=\tfrac{S}{S+T}) + \frac{1}{R^4}G^{(2)}_{\text{closed}}(z=\tfrac{S}{S+T}) + \ldots$$

✓

This implies

$$G^{(2)}_{\text{closed}}(z=\tfrac{S}{S+T}) = \frac{1}{2}\left(G^{(1)}_{\text{closed}}(z=\tfrac{S}{S+T})\right)^2$$

✓

Final result to all orders in $S/R$:

$$\lim_{S,T,R\to\infty} A^{\text{AdS}}(S,T) = \left(\lim_{S,T\to\infty} A^{\text{flat}}(S,T)\right) e^{-\frac{S^2}{R^2}F_3\left(\frac{S}{T}\right)}$$

# Summary: High energy limit

We compared $A(S,T)$ to classical computation a la Gross & Mende.

- Relation to worldsheet action agnostic to fermions and prefactors
- $A(S,T)$ fixed to all orders in $S/R$

$$\lim_{S,T,R\to\infty} A^{\mathrm{AdS}}(S,T) = \left(\lim_{S,T\to\infty} A^{\mathrm{flat}}(S,T)\right) e^{-\varepsilon_{\mathrm{open/closed}}(S,T)}$$

- The exponents (weight 3 SVMPLs) for open and closed strings satisfy the expected relation:

$$\varepsilon_{\mathrm{open}}(S,T) = \frac{1}{2}\varepsilon_{\mathrm{closed}}(4S,4T)$$

✓

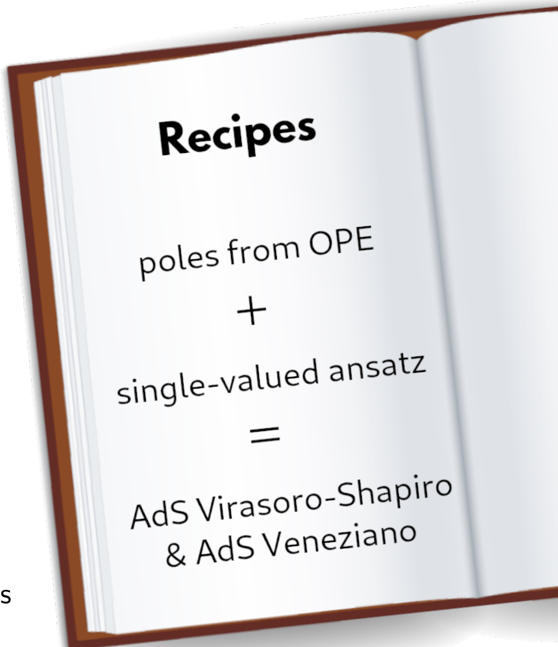
# Summary

STRING AMPLITUDE
SHOPPING LIST

- PARTIAL WAVE EXPANSION
- REGGE BOUNDEDNESS
- WORLDSHEET INTEGRAL

Checks:

- Low energy expansion
- OPE data for massive strings
- High energy limit



**Recipes**

poles from OPE

$+$

single-valued ansatz

$=$

AdS Virasoro-Shapiro
& AdS Veneziano

# Future directions

- Dimensions for massive open string operators from integrability?
- Other AdS backgrounds, e.g. type IIA on $AdS_4 \times CP^3$ / ABJM
- Go beyond the small curvature expansion?
- Compute AdS string amplitudes directly from string theory?

Thank you!