All Five-point Kaluza–Klein Correlators and Hidden 8d Symmetry in $\mathrm{AdS}_5 \times \mathrm{S}^3$

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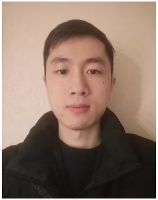
collaboration with



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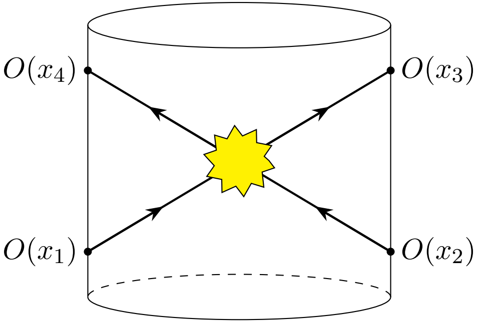
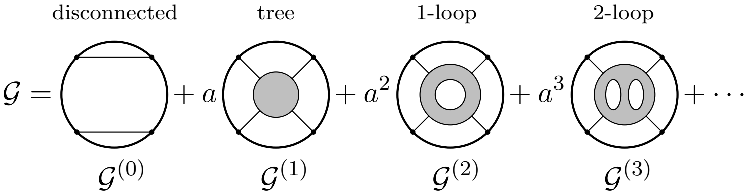
Holographic correlators

An important class of observables in the context of AdS/CFT.



- Bulk: perturbative scattering of particles in AdS.
- Boundary: correlation of local operators in strongly-coupled CFT.

Perturbative expansion in the bulk



+(stringy corrections)

Traditional approach: Witten diagrams.

- Bulk-to-bulk propagators are non-trivial functions of AdS invariant distance.
- Integrals inside AdS is more complicated.
- Tremendous amount of interaction vertices.

About Kaluza–Klein modes

- Typical AdS/CFT models: $\mathrm{AdS}_n \times$ (internal space).
- Particles can move not only purely in AdS, but also in the internal space.
- Boundary: they correspond to a sequence of 1/2-BPS operators, indexed by a Kaluza–Klein charge.

The resulting bulk interaction is **RICH** and **DIFFICULT**.

↑ organizing principles

Modern approach

- Modern approach to the computation of holographic correlators is realized by **BOOTSTRAP**.
- This is pioneered by the work of Rastelli and Zhou, which computes all $\langle O_p O_q O_r O_s \rangle$ in $\mathrm{AdS}_5 \times \mathrm{S}^5$ [‘16, ‘17].
  - O_2: (scalar super-partner of) graviton in AdS_5.
  - $O_{p>2}$: its KK compactifications on S^5.
- With the constraints from superconformal symmetry [Nirschl, Osborn, ’04]

$$\langle O_p O_q O_r O_s \rangle = \mathcal{G}^{\mathrm{free}}_{pqrs} + R\, \mathcal{H}_{pqrs}.$$

  - R: appropriate kinematic factors.
  - $\mathcal{G}^{\mathrm{free}}_{pqrs}$: determined by mean field theory.
  - All dynamics is encoded in the reduced correlator $\mathcal{H}_{pqrs}$.
- Bootstrap leads to **a single unified formula for all KK correlators!**

Modern approach: Mellin space

For conformal correlators it is useful to introduce
Mellin amplitudes [Mack, '09][Penedones, '11]

$$\langle O_{p_1}(x_1)\cdots O_{p_n}(x_n)\rangle_{\text{connected}} = \int [\mathrm{d}\gamma]\,\mathcal{A}_{\{p\}}(\gamma)\prod_{i<j}\frac{\Gamma(\gamma_{ij})}{((x_i-x_j)^2)^{\gamma_{ij}}}.$$

γ_{ij}: Mellin variables, constrained by

$$\gamma_{ij}=\gamma_{ji},\qquad \sum_{j=1}^{n}\gamma_{ij}=0,\qquad \gamma_{ii}\equiv -p_i.$$

- $p_i + p_j - 2\gamma_{ij}$ resembles Mandelstam variables.
- Locations of poles $\Leftrightarrow$ twists of operators in the OPE.
- Factorize on poles, like scattering amplitudes.
- Much simpler structure: e.g., for contact diagrams $\mathcal{M}$ = polynomial.

Modern approach: bootstrap

- At 4 points, study Mellin amplitude for reduced correlator

$$\mathcal{H}_{pqrs} = \int [\mathrm{d}\delta]\, \mathcal{M}_{pqrs}(\rho) \prod_{i<j} \frac{\Gamma(\delta_{ij})}{((x_i - x_i)^2)^{\delta_{ij}}}$$

($\delta_{ii} \equiv 2 - p_i$ due to shifts caused by $\mathcal{R}$)

- Set up ansatz at tree level in $\mathrm{AdS}_5 \times \mathrm{S}^5$

R-symmetry factor

$$\mathcal{M}_{pqrs} = \sum_{i,j,k} \frac{a_{ijk}\sigma^i \tau^i}{(s - s_{\mathrm{m}} + 2i)(t - t_{\mathrm{m}} + 2j)(\tilde{u} - \tilde{u}_{\mathrm{m}} + 2k)}$$

Range determined by selection rules

- a_{ijk} solved by symmetries, flat space limit, asymptotics, polynomial behavior or residues, **FACTORIZATION**, etc.

Use of KK correlators: hidden structures

- 4-point KK correlators possess a hidden 10d conformal symmetry [Caron-Huot, Trinh, '18].
- Relation between reduced correlators

$$\mathcal{H}_{pqrs} = \mathcal{D}_{pqrs}\mathcal{H}_{2222}.$$

- Alternatively, expressed in terms of a generating function [Alday, Zhou, '19]

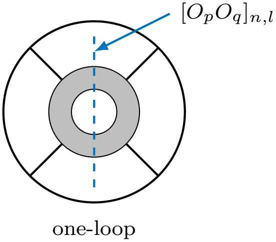
$$\mathbf{H}((x_i - x_j)^2, y_i \cdot y_j) = \mathcal{H}_{2222}((x_i - x_j)^2 + y_i \cdot y_j).$$

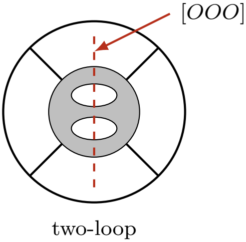
Expand and extract all R-symmetry structures for $\mathcal{H}_{pqrs}$.

Q1: Does the hidden structure extend to higher points?

Use of KK correlators: loop corrections in AdS

- Computations of loop-level scattering are realized by UNITARITY + BOOTSTRAP [Aharony, et al, '16].
- Lower-loop CFT data ⇒ higher-loop singularities.





- For $\mathcal{H}_{2222}^{2-\text{loop}}$, a structural observation helps avoid the input of $[OOO]$ [Huang, EYY, '21][Drummond, Paul, '22].

Q2: CFT data for triple-particle operators?

In this work, we bootstrap

all five-point Kaluza–Klein correlators

Model to be studied

- Instead of gravitons, we choose to study gluons.
- Can be realized on an $AdS_5 \times S^3$ background [Alday, et al, '21]
  - by inserting probe D7 branes in $AdS_5 \times S^5$.
  - by D3 branes probing F-theory singularities.
- $\mathcal{N} = 2$ SCFT on the boundary.
  We only focus on the gluon sector.
- (Scalar) gluons + Kaluza–Klein tower

$$\mathcal{O}_p^I(x; v, \bar{v}) \equiv \mathcal{O}_p^{I;\alpha_1 \ldots \alpha_p;\beta_1 \ldots \beta_{p-1}} v_{\alpha_1} \cdots v_{\alpha_p} \bar{v}_{\beta_1} \cdots \bar{v}_{\beta_{p-2}}.$$

  I - gauge (bulk). v - $SU_R(2)$. $\bar{v}$ - $SU_L(2)$.

  - $x_{ij}^2 \equiv (x_i - x_j)^2$, $v_{ij} \equiv \epsilon_{\alpha\beta} v_i^\alpha v_j^\beta$, $\bar{v}_{ij} \equiv \epsilon_{\alpha\beta} \bar{v}_i^\alpha \bar{v}_j^\beta$.

an easier environment to make new observations

A first look at the structure

- Decomposition according to traces of color generators

$$\langle O_{p_1}(x_1)\cdots O_{p_n}(x_n)\rangle = \sum_{\sigma\in \mathrm{S}_n/\mathrm{Z}_2} \mathrm{tr}\left[T^{I_{\sigma(1)}}\cdots T^{I_{\sigma(n)}}\right] G[\sigma] + \cdots .$$

  We focus on $G_n \equiv G[12\cdots n]$.

- Complexity of R-symmetry structures is tied to extremality (assuming $O_{p_n}(x_n)$ has the largest twist)

$$2\mathcal{E} = p_1 + p_2 + \cdots + p_{n-1} - p_n .$$

- Correlators are non-vanishing only for $\mathcal{E} \geq n-2$.
  (useful selection rules later on)
- Past study suggests
  it is a good strategy to focus on fixed $\mathcal{E}$ at a time.
  So we work with $\mathcal{E} = 3, 4, 5, \ldots$ for five-point correlators.

Four-points vs higher-points

- At four points, many of the results in graviton scattering have their analogues in gluon scattering.
- A hidden 8d conformal symmetry exists. [Alday, et al, '21]
- Superconformal symmetry constrains that

$$G_4 = G_4^{\text{free}} + \underbrace{\left(V_{1234}\, x_{13}^2 x_{24}^2 + V_{1342}\, x_{14}^2 x_{23}^2 + V_{1423}\, x_{12}^2 x_{34}^2\right)}_{R_{1234}} H_4,$$

  with $V_{1234} \equiv v_{12} v_{23} v_{34} v_{41}$. R_{1234} is permutation invariant.
- No higher-point analogue is known.
  Higher-point correlators with low KK charges were studied.
  [Alday et al, '22 '23][Cao et al, '23 '24] (explicitly $\leq$7-pt; in principle higher)

Q3: Write $G_{n>4}$ in a way trivializing superconformal?

We obtain a **unified** formula at five points.

Q1: Does the hidden structure extend to higher points?

Yes at five points.

Q2: CFT data for triple-particle operators?

Leave for future work.

Q3: Write $G_{n>4}$ in a way trivializing superconformal?

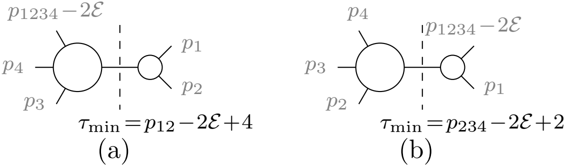
Interesting observations.

OPE selection rules

- Mellin amplitude has poles at

$$\gamma_{12} = \frac{p_1 + p_2 - \tau}{2} - k.$$

- Bounded from below by $\Gamma(\gamma_{12})$ in the Mellin transform.
- Upper bound (or min of τ) set by the min extremality of sub-amplitudes (consider exchange of O_τ)



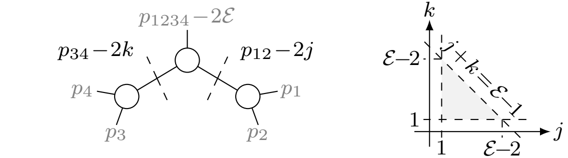
- Allowed poles in a single channel

$$\begin{aligned} &\text{channel (12)}: \ \gamma_{12} - j, && j = 1, 2, \ldots, \mathcal{E} - 2, \\ &\text{channel (15)}: \ \gamma_{15} - p_1 + k, && k = 1, 2, \ldots, \mathcal{E} - 1. \end{aligned}$$

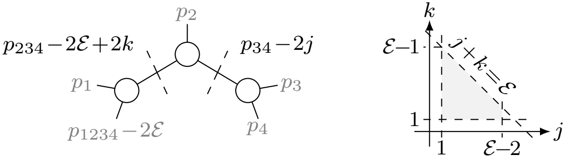
OPE selection rules

- For consecutive OPEs, a further constraint from the min extremality of the middle sub-amplitude

(A)



(B)



- There are also other component fields in the vector multiplet of O (discussed later).
  Their twists are greater than O, hence do not affect the above counting.

Ansatz

- At 5 points there are five independent Mellin variables. We choose them to be $\{\gamma_{12}, \gamma_{23}, \gamma_{34}, \gamma_{45}, \gamma_{15}\}$.
- Kinematic bases
  - Simultaneous poles

$$(\mathrm{A}) : \frac{\{1, \gamma_{23}, \gamma_{45}, \gamma_{15}\}}{(\gamma_{12}-j)(\gamma_{34}-k)}, \quad j,k \geq 1,\ j+k \leq \mathcal{E}-1,$$

$$(\mathrm{B}) : \frac{\{1, \gamma_{12}, \gamma_{23}, \gamma_{45}\}}{(\gamma_{34}-1)(\gamma_{15}-p_1+k)}, \quad j,k \geq 1,\ j+k \leq \mathcal{E},$$

  - Single poles

$$(\mathrm{a}) : \frac{\{1\}}{\gamma_{12}-j}, \qquad j = 1, 2, \ldots, \mathcal{E}-2,$$

$$(\mathrm{b}) : \frac{\{1\}}{\gamma_{15}-p_1+k}, \quad k = 1, 2, \ldots, \mathcal{E}-1.$$

Name the whole list $\mathcal{K}$ (collecting all channels).

Ansatz

- R/L-structure bases: work in a specific frame

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \; v_3 = \begin{pmatrix} 1 \\ z_1 \end{pmatrix}, \; v_4 = \begin{pmatrix} 1 \\ z_2 \end{pmatrix}, \; v_5 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

  Similarly for $\bar{v}$'s (with z replaced by w).

  - In principle, can work out compatible R/L-structures for each kinematic basis element, but this is not economic.
  - In practice, simply list all R/L-structures for a given $\mathcal{E}$

$$z_1^m z_2^n w_1^i w_2^j, \qquad m + n \leq \mathcal{E} - 1, \quad i + j \leq \mathcal{E} - 3$$

  Name the whole list $\mathcal{I}$.

- Full ansatz: take the direct product

$$\mathcal{K} \otimes \mathcal{I}$$

  and make linear combinations with unknown coefficients.

Operators in the exchange and factorizations

- We solve the ansatz solely by studying FACTORIZATIONS.
- Operators that can be exchanged

operator	$\mathcal{O}_p$	$\mathcal{J}_p^\mu$	$\mathcal{F}_p$
twist	p	p	$p+2$
Lorentz spin	0	1	0
$\mathrm{SU}(2)_{\mathrm{R}}$ spin	$\frac{p}{2}$	$\frac{p}{2}-1$	$\frac{p}{2}-2$
$\mathrm{SU}(2)_{\mathrm{L}}$ spin	$\frac{p}{2}-1$	$\frac{p}{2}-1$	$\frac{p}{2}-1$

- Need correlators at lower points

$$\langle JOO\rangle, \quad \langle FOO\rangle, \quad \langle JOOO\rangle, \quad \langle FOOO\rangle.$$

- We work out these correlators from $\langle OOO\rangle$ and $\langle OOOO\rangle$ using analytic superspace techniques.

Operators in the exchange and factorizations

- Factorization on scalar (O or F) poles (assuming point $1, 2, \ldots, k$ are on the left) [Fitzpatrick, et al, '11]

$$M \sim \frac{1}{\gamma_{i,i+1} - \frac{p_i + p_{i+1} - p}{2} + m} \frac{\Gamma(p)\, m!}{(p-1)_m} M_{L,m} M_{R,m},$$

$$M_{L,m} = \sum_{\substack{i_{ab} \geq 0, \\ \sum i_{ab} = m}} M_L(\gamma_{ab} + i_{ab}) \prod_{1 \leq a < b \leq k} \frac{(\gamma_{ab})_{i_{ab}}}{i_{ab}!}.$$

- Similar factorization for spin one (J) has also been studied before [Goncalves, et al, '14].
- A pole can receive contributions from primaries ($m = 0$) and conformal descendants of other primaries ($m > 0$). Need to sum them up when comparing the ansatz.
- Gluing $\mathrm{SU}(2)_{\mathrm{R}}$ and $\mathrm{SU}(2)_{\mathrm{L}}$ structures.

With this bootstrap procedure, we obtain formulas for

$$\langle O_p O_q O_r O_s O_{p+q+r+s-2\mathcal{E}} \rangle$$

with $\mathcal{E} = 3, 4, 5, 6$, respectively.

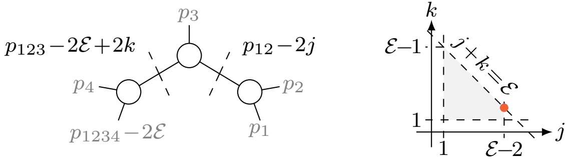
The direct computational result looks cumbersome...

... yet there are STRUCTURES buried deep inside!

Hints for a unified formula

- Choose some simultaneous poles sitting at the corner of the poles grid, e.g., at $\gamma_{12} = \mathcal{E} - 2$ and $\gamma_{45} = p_4 - 2$

(B)



- Keeping track of the change as $\mathcal{E}$ varies, this term reads

$$A_{\{p_i\}} \supset \frac{t_{12}^{\mathcal{E}-3}}{(\mathcal{E}-3)!} \frac{t_{15}^{p_1-\mathcal{E}+1}}{(p_1-\mathcal{E}+1)!} \frac{t_{25}^{p_2-\mathcal{E}+1}}{(p_2-\mathcal{E}+1)!} \frac{t_{35}^{p_3-2}}{(p_3-2)!} \frac{t_{45}^{p_4-2}}{(p_4-2)!} \times \frac{(p_4-2)\, v_{12}^2\, v_{34}\, v_{45}\, v_{53}\, \gamma_{23}}{(\gamma_{12}-\mathcal{E}+2)(\gamma_{45}-p_4+2)},$$

Here $t_{ij} \equiv v_{ij}\bar{v}_{ij}$.

- Factors in blue suggest a "Mellin" transformation on S^3.

[Vieira, Aprile, '20]

Hints for a unified formula

- Generalized Mellin amplitude (Mellin transform also on S^3)

$$\mathcal{F}_{\{p\}} = \sum_{n_{ij}} \int [\mathrm{d}\gamma_{ij}]\, \widetilde{\mathcal{M}}_{\{p\}}(\gamma, n) \prod_{i<j} \frac{t_{ij}^{n_{ij}}}{\Gamma(1+n_{ij})} \frac{\Gamma(\gamma_{ij})}{((x_i - x_i)^2)^{\gamma_{ij}}},$$

  Summation is truncated by $1/\Gamma(1+n_{ij})$.
  The previous expression as a term in this summation.
- If $\widetilde{M}_{\{p\}}$ is independent of $\{p\}$, then we can construct a generating function ($\rho_{ij} \equiv \gamma_{ij} - n_{ij}$)

$$\mathbf{F} = \sum_{p_i=0}^{\infty} \mathcal{F}_{\{p\}} = \int [\mathrm{d}\rho]\, \widetilde{\mathcal{M}}(\rho_{ij}) \prod_{i<j} \frac{\Gamma(\rho_{ij})}{(x_{ij}^2 - t_{ij})^{\rho_{ij}}}$$

- A manifestation of hidden 8d structures.

Hints for a unified formula

- At 4 points

$$G_4 = G_4^{\text{free}} + \underbrace{\left(V_{1234}\, x_{13}^2 x_{24}^2 + V_{1342}\, x_{14}^2 x_{23}^2 + V_{1423}\, x_{12}^2 x_{34}^2\right)}_{R_{1234}} H_4,$$

$$\mathbf{A}_4 = \sum_{n_{ij}} \left(\widehat{R}_{1234} \circ \widetilde{M}_4\right) \prod_{i<j} \frac{t_{ij}^{n_{ij}}}{\Gamma(1+n_{ij})}, \qquad \widetilde{M}_4 = \frac{1}{(\rho_{12}-1)(\rho_{14}-1)}.$$

- The Mellin space operator

$$\widehat{R}_{1234} = V_{1234}\, \hat{\gamma}_{13}\hat{\gamma}_{24} + V_{1342}\, \hat{\gamma}_{14}\hat{\gamma}_{23} + V_{1423}\, \hat{\gamma}_{12}\hat{\gamma}_{34}\,,$$

acts as multiplication and shift

$$\hat{\gamma}_{ij} \circ F(\gamma_{ij}, n_{ij}) = \gamma_{ij}\, F(\gamma_{ij}+1, n_{ij})\,,$$

and is again permutation invariant.

A dozen of days later ...

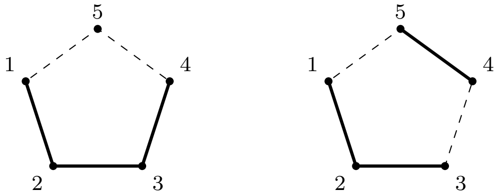
Please be kind NOT to ask me what happened in the middle :-)

Formula (for planar correlators)

$$\mathbf{A}_5 = \widehat{R}^{(1)} \circ \widetilde{M}_5^{(1)} + \widehat{R}^{(2)} \circ \widetilde{M}_5^{(2)} + (\text{cyclic})\,,$$

where

$$\widetilde{M}_5^{(1)} = -\frac{1}{5\,(\rho_{12}-1)(\rho_{23}-1)(\rho_{34}-1)}\,,$$
$$\widetilde{M}_5^{(2)} = -\frac{2}{5\,(\rho_{12}-1)(\rho_{23}-1)(\rho_{45}-1)}\,,$$



Formula (for planar correlators)

$$\widehat{R}^{(1)} = \widehat{R}_{1234,5} + \widehat{R}_{2345,1} + \widehat{R}_{3451,2} + \widehat{R}_{4512,3} + \widehat{R}_{5123,4}$$
$$+ (\mathrm{v}_{13,5}\,\hat{n}_{13} + \mathrm{v}_{23,5}\,\hat{n}_{23} + \mathrm{v}_{14,5}\,\hat{n}_{14} + \mathrm{v}_{24,5}\,\hat{n}_{24})\widehat{R}_{1234}$$
$$+ (\mathrm{v}_{13,4}\,\hat{n}_{13} + \mathrm{v}_{23,4}\,\hat{n}_{23} + \mathrm{v}_{53,4}\,\hat{n}_{53})\widehat{R}_{1235}$$
$$+ (\mathrm{v}_{23,1}\,\hat{n}_{23} + \mathrm{v}_{24,1}\,\hat{n}_{24} + \mathrm{v}_{25,1}\,\hat{n}_{25})\widehat{R}_{2345}\,,$$

$$\widehat{R}^{(2)} = \widehat{R}_{4512,3} + \widehat{R}_{4513,2} + \widehat{R}_{4521,3} + \widehat{R}_{4523,1} + \widehat{R}_{4531,2}$$
$$+ \widehat{R}_{4532,1} + (\mathrm{v}_{14,5}\,\hat{n}_{14} + \mathrm{v}_{24,5}\,\hat{n}_{24} + \mathrm{v}_{34,5}\,\hat{n}_{34})\widehat{R}_{1234}$$
$$+ (\mathrm{v}_{51,4}\,\hat{n}_{51} + \mathrm{v}_{52,4}\,\hat{n}_{52} + \mathrm{v}_{53,4}\,\hat{n}_{53})\widehat{R}_{1235}\,.$$

Here $v_{ij,k} \equiv v_{ik}v_{jk}/v_{ij}$, and $\hat{n}_{ij} \circ F(\gamma_{ij}, n_{ij}) = n_{ij}\, F(\gamma_{ij}, n_{ij} - 1)$.

An additional elementary **five-label** $\widehat{R}$ operator

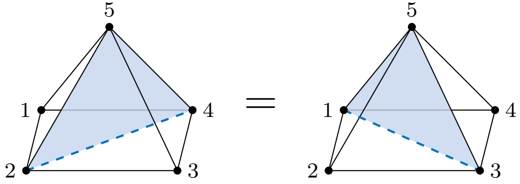
$$\widehat{R}_{1234,5} = V_{12345}\,\hat{\gamma}_{14}\hat{\gamma}_{25}\hat{\gamma}_{35} + V_{12354}\,\hat{\gamma}_{34}\hat{\gamma}_{15}\hat{\gamma}_{25}$$
$$+ V_{12534}\,\hat{\gamma}_{23}\hat{\gamma}_{15}\hat{\gamma}_{45} + V_{15234}\,\hat{\gamma}_{12}\hat{\gamma}_{35}\hat{\gamma}_{45}$$

Comments on the five-label $\widehat{R}$

- $\widehat{R}_{1234,5}$ enjoys cyclic permutation invariance in (1234), and switches sign under reflection.
- It can be (non-uniquely) decomposed onto four-label $\widehat{R}$'s

$$
\begin{aligned}
\widehat{R}_{1234,5} &= v_{24,1}\hat{\gamma}_{15}\,\widehat{R}_{2345} + v_{42,3}\hat{\gamma}_{35}\,\widehat{R}_{1245}, \\
&= v_{13,4}\hat{\gamma}_{45}\,\widehat{R}_{1345} + v_{31,2}\hat{\gamma}_{25}\,\widehat{R}_{1235}.
\end{aligned}
$$

- Geometric intuition: cutting a pyramid



Why such structure emerges?

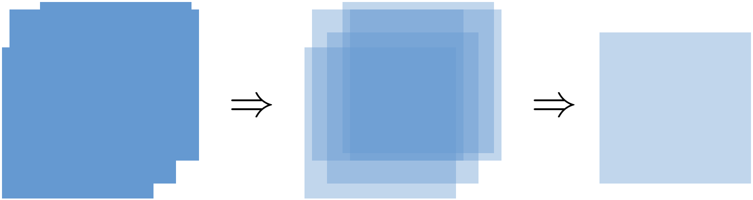
Outlook

- CFT data
- Higher points
- Graviton scattering
- Weak coupling

Thank you very much!

Questions & comments are welcome.

Pain in reversing engineering ...



A MASS. A pile of simpler stuff? The ATOM.