Constructive Holography

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Message in a nutshell

Collective field theory provides a constructive approach to the AdS/CFT correspondence. (Jevicki, Sakita, **NPB** 165 (1980) 511)

Basic claim: Holography is accomplished by a change to gauge invariant field variables in the CFT. A change of spacetime coordinates is needed to give the bulk interpretation of the collective field theory.(Das and Jevicki, Phys. Rev. D **68** (2003), 044011)

Starting from the CFT and carrying out these two steps, one obtains the higher dimensional gravitational theory.

Holography is demonstrated by giving a holographic map constructed by:

1. Reducing gravity to physical and independent degrees of freedom.
2. Reducing CFT to its independent degrees of freedom.
3. Identifying the complete set of degrees of freedom of CFT with those of gravity.

The mapping reproduces general expectations of holography including loop expansion parameter $1/N$, bulk reconstruction and a complete set of degrees of freedom reside on the boundary.

Comments before we begin

We work entirely at the leading order at large N. Thus we aim to reproduce the linearized gravity.

Correlators have already been compared. It would be nice to show that the collective field theory vertices agree with those of gravity. Completely guage fixed versions of the dual gravity are not yet readily available.

We work entirely around empty AdS.

Outline

What is a collective field?

Why use collective fields?

Matching degrees of freedom (AdS_4/CFT_3)

Holographic map between collective field theory and gravity (AdS_4/CFT_3)

Redundancy and holography (AdS_4/CFT_3)

Testing the holographic map (AdS_4/CFT_3)

Conclusions.

What is a collective field?

Collective Field Theory

- is a framework to describe dynamics of large number of interacting particles/fields.
- focuses on collective excitations (degrees of freedom) describing overall behaviour of system.
- is useful when the individual behaviour of particles is less important than their collective effects.

The collective fields are given by the general set of gauge invariant operators.

Essence of the quantum collective field method: consider most general (over complete) set of commuting operators and explicitly perform change of variables to the new set (collective field).

What is a collective field? Examples.

O(N) vector model: Original field is O(N) vector $\phi^a(x^\mu)$. Collective field is a bilocal.

$$\sigma(x^\mu, y^\mu) = \sum_a \phi^a(x^\mu)\phi^a(y^\mu)$$

$$\int [D\phi^a]\, e^{iS} \rightarrow \int [D\sigma]\, e^{iS_{\rm eff}}$$

Matrix model: Fields are adjoint scalars $M_I(x^\mu)_{ab}$. Collective fields are k-local.

$$\sigma_k(x_1^\mu, x_2^\mu, \cdots, x_k^\mu) = {\rm Tr}(M_{I_1}(x_1^\mu)M_{I_2}(x_2^\mu)\cdots M_{I_k}(x_k^\mu))$$

$$\int \prod_I [DM_I]\, e^{iS} \rightarrow \int \prod_k [D\sigma_k]\, e^{iS_{\rm eff}}$$

Yang-Mills theory: Gauge fields, adjoint scalars. Collective fields are k-local.

$$\sigma_k(x_1^\mu, x_2^\mu, \cdots, x_k^\mu) = {\rm Tr}(M_{I_1}(x_1^\mu)W(x_1, x_2)M_{I_2}(x_2^\mu)W(x_2, x_3)\cdots M_{I_k}(x_k^\mu)W(x_k, x_1))$$

$$\int \prod_I [DM_I]\, e^{iS} \rightarrow \int \prod_k [D\sigma_k]\, e^{iS_{\rm eff}}$$

What is a collective field?

A single collective field packages all possible single trace primary operators that can be constructed from the constituent fields.

The bilocal

$$\sigma(x^\mu, y^\mu) \;=\; \sum_a \phi^a(x^\mu)\phi^a(y^\mu)$$

in $d = 3$ packages a scalar

$$O_{\Delta=1}(t, \vec{x}) = \sum_{a=1}^{N} \phi^a(t, \vec{x})\phi^a(t, \vec{x})$$

and a tower of conserved currents

$$J_{\mu_1\mu_2\cdots\mu_{2s}}(t, \vec{x})\alpha^{\mu_1}\alpha^{\mu_2}\cdots\alpha^{\mu_{2s}} = \sum_{a=1}^{N}\sum_{k=0}^{2s} \frac{(-1)^k \;:(\alpha\cdot\partial)^{2s-k}\phi^a\,(\alpha\cdot\partial)^k\phi^a:}{k!(2s-k)!\Gamma(k+\frac{1}{2})\Gamma(2s-k+\frac{1}{2})}$$

The standard holographic dictionary associates a bulk gravity field to each single trace primary operator $\Rightarrow$ collective fields package the complete set of gravity fields.

Collective Field Theory

Phrase the dynamics in terms of the collective field

$$\sigma(t_1,\vec{x}_1,t_2,\vec{x}_2)=\sum_{a=1}^{N}\phi^a(t_1,\vec{x}_1)\phi^a(t_2,\vec{x}_2) \qquad \sigma(t,\vec{x}_1,\vec{x}_2)=\sum_{a=1}^{N}\phi^a(t,\vec{x}_1)\phi^a(t,\vec{x}_2)$$

At the path integral level

$$\int[d\phi^a(t,\vec{x})]e^{iS}\cdots \quad = \quad \int[d\sigma(t_1,\vec{x}_1,t_2,\vec{x}_2)]J[\sigma]e^{iS}\cdots$$

$J[\sigma]$ is fixed so that the bilocal theory has the correct Schwinger-Dyson equations. For an equal time quantization using a Hamiltonian, use the chain rule

$$\begin{aligned}\pi^a(t,\vec{x})=\frac{1}{i}\frac{\delta}{\delta\phi^a(t,\vec{x})}&=\int d\vec{x}_1\int d\vec{x}_2\frac{\delta\sigma(t,\vec{x}_1,\vec{x}_2)}{\delta\phi^a(t,\vec{x})}\ \frac{1}{i}\frac{\delta}{\delta\sigma(t,\vec{x}_1,\vec{x}_2)}\\&=\int d\vec{x}_1\int d\vec{x}_2\frac{\delta\sigma(t,\vec{x}_1,\vec{x}_2)}{\delta\phi^a(t,\vec{x})}\ \Pi(t,\vec{x}_1,\vec{x}_2)\end{aligned}$$

Requiring the Hamiltonian $J^{\frac{1}{2}}HJ^{-\frac{1}{2}}$ is hermittian fixes the measure.

Redundancy in Collective Field Theory

We treat all degrees of freedom in the collective field as independent.

$$\langle \cdots \rangle = \int d\sigma(x_1^\mu, x_2^\mu) e^{iS} \cdots$$

$$[\Pi(t,\vec{x}_1,\vec{x}_2), \sigma(t,\vec{y}_1,\vec{y}_2)] = -i\delta(\vec{x}_1-\vec{y}_1)\delta(\vec{x}_2-\vec{y}_2) \qquad \Pi(t,\vec{x}_1,\vec{x}_2) = \frac{1}{i}\frac{\delta}{\delta\sigma(t,\vec{x}_1,\vec{x}_2)}$$

This is a very redundant description compared to the original description

$$\langle \cdots \rangle = \int d\phi^a(x^\mu) e^{iS} \cdots$$

$$[\pi^a(t,\vec{x}), \phi^b(t,\vec{y})] = -i\delta^{ab}\delta(\vec{x}-\vec{y}) \qquad \pi^a(t,\vec{x}) = \frac{1}{i}\frac{\delta}{\delta\phi^a(t,\vec{x})}$$

One independent degree of freedom at each point $(\vec{x}_1, \vec{x}_2)$ versus N independent (up to singlet condition) degrees of freedom at each $\vec{x}$.

$$\sigma(t,\vec{x},\vec{y}) \quad \text{versus} \quad \phi^a(t,\vec{x})$$

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Why use collective fields?

The loop expansion parameter of the original CFT is $\hbar$. The loop expansion parameter of the dual gravity is $\frac{1}{N}$.

After changing to invariant (collective) variables the loop expansion parameter is $1/N$ matching the loop expansion parameter of the dual gravity. (general feature of collective field theory)

Why invariant variables are naturally relevant for large N: consider the bilocal field $\sigma(x_1, x_2)$ of the O(N) vector model

$$\sigma(x_1, x_2) \quad = \quad \frac{1}{N} \sum_{a=1}^{N} \phi^a(x_1)\phi^a(x_2)$$

$=$ mean of N identically distributed independent random variables $\Rightarrow$ by the central limit theorem $\sigma(x_1, x_2)$ approaches a definite classical field at large N.

$1/N$ as loop expansion parameter

To solve QFT "all" we need to do is evaluate a complicated integral.

$$\int d\phi^a e^{-\frac{1}{\hbar}S(\phi^a)} \qquad a = 1, \cdots, N$$

Hard when $N \to \infty$, but if theory is $O(N)$ symmetric $\Rightarrow$action is $O(N)$ invariant.

Suppose the ϕ^a are in vector rep of $O(N)$. Then S is a function of $\sigma = \phi^a\phi^a$, the unique invariant. One integration variable and not N - much simpler!

$$\int d\sigma e^{-N\tilde{S}(\sigma)} \quad = \quad \int d\sigma e^{-\frac{1}{\left(\frac{1}{N}\right)}\tilde{S}(\sigma)}$$

N appears because we integrate over $N-1$ variables, leaving a single σ integral. $O(N)$ implies symmetry each integral gives identical contribution. Saddle point approximation gives loop expansion with $1/N$ as loop counting parameter.

The number of collective field variables has no explicit N dependence and we can factor a power of N in front of the action. (ϕ^a versus $\sigma = \sum_a \phi^a\phi^a$)

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Counting degrees of freedom

Counting in gravity: Work in lightcone gauge (AdS_{d+1}, $d+1=4$) $A^{+\mu_2\cdots\mu_{2s}}=0$. All components $A^{-\mu_2\cdots\mu_{2s}}$ are determined by constraints. Dynamical fields are X, Z polarizations: $A^{XZXZ\cdots ZZ}$. Gauge field is symmetric and traceless, $\Rightarrow$ **two independent** physical degrees of freedom at each spin.

Metsaev **NPB** 563 (1999) 295 hep-th/9906217

Counting in CFT: The spinning currents ($d=2+1$) are symmetric, traceless, conserved rank $2s$ tensors $J_{\mu_1\cdots\mu_{2s}}$.

There are $\frac{(2s+1)(2s+2)}{2}$ symmetric rank $2s$ tensors.

There are $4s+1$ symmetric, traceless rank $2s$ tensors.

There are **two independent** symmetric, traceless, conserved rank $2s$ tensors.

$\Rightarrow$ **number of independent components of the spinning primary match the number of physical and independent components of the gauge field.**

Reduce gravity to independent and physical fields. Reduce CFT to its independent fields. To construct holographic map, identify physical degrees of freedom.

Symmetries in the reduced theory

Higher spin currents $J_s^{\mu_1\mu_2\cdots\mu_s}(x^\nu)$, have $\Delta = s+1$, are traceless and conserved

$$\partial_\mu J_s^{\mu\mu_2\cdots\mu_s}(x^\nu) \;=\; 0 \qquad \eta_{\mu\nu} J_s^{\mu\nu\mu_3\cdots\mu_s}(x^\nu) \;=\; 0$$

Represent h.s. current as $\quad |J_s(t,\vec{x},a^\mu)\rangle \;=\; J_s^{\mu_1\mu_2\cdots\mu_s}(x^\nu)a_{\mu_1}\cdots a_{\mu_s}|0\rangle$

$$[\bar{a}^\mu, a^\nu] \;=\; \eta^{\mu\nu} \qquad \mu,\nu = 0,1,2 \qquad \bar{a}^\mu|0\rangle \;=\; 0$$

Conservation and traceless conditions are now

$$\bar{a}^\nu\partial_\nu|J_s(t,\vec{x},a^\mu)\rangle \;=\; 0 \qquad \bar{a}^\nu\bar{a}_\nu|J_s(t,\vec{x},a^\mu)\rangle \;=\; 0$$

Conservation eliminates one polarization. Eliminate $+$ polarization to find

$$|J_{(s)}\rangle \;=\; \exp\left(-a^+\left[\frac{\bar{a}^+\partial^- + \bar{a}^b\partial^b}{\partial^+}\right]\right)|i_{(s)}\rangle \equiv \mathcal{P}|i_{(s)}\rangle$$

$$|\text{reduced state}\rangle \;=\; |i_{(s)}\rangle \;=\; j_{(s)}^{i_1 i_2\cdots i_s} a_{i_1}a_{i_2}\cdots a_{i_s}|0\rangle \qquad i_k = -, b$$

O acting on the original currents $|J_{(s)}\rangle$ becomes $\tilde{O} = \mathcal{P}^{-1}\,O\,\mathcal{P}$.

Metsaev **NPB** 563 (1999) 295 hep-th/9906217

Symmetries in the reduced theory

Computing $\tilde{O} = \mathcal{P}^{-1}\, O\, \mathcal{P}$ for the symmetry generators we find

$$\tilde{J}^{+-} \;=\; \mathcal{P}^{-1} J^{+-} \mathcal{P} \;=\; x^{+} \frac{\partial}{\partial x^{+}} - x^{-} \frac{\partial}{\partial x^{-}} - a^{-} \frac{\partial}{\partial a^{-}}$$

$$\tilde{J}^{+i} \;=\; \mathcal{P}^{-1} J^{+i} \mathcal{P} \;=\; x^{+} \frac{\partial}{\partial x^{i}} - x^{i} \frac{\partial}{\partial x^{-}} - a^{i} \frac{\partial}{\partial a^{-}}$$

$$\tilde{J}^{-i} \;=\; \mathcal{P}^{-1} J^{-i} \mathcal{P} \;=\; x^{-} \frac{\partial}{\partial x^{i}} - x^{i} \frac{\partial}{\partial x^{+}} + a^{-} \frac{\partial}{\partial a^{i}} + a^{i} \frac{\bar{a}^{b} \partial^{b}}{\partial^{+}}$$

and

$$\tilde{J}^{ij} \;=\; \mathcal{P}^{-1} J^{ij} \mathcal{P} \;=\; x^{i} \partial^{j} - x^{j} \partial^{i} + a^{i} \bar{a}^{j} - a^{j} \bar{a}^{i}$$

The $+$ polarizations have indeed been eliminated. (No $\frac{\partial}{\partial a^{+}}$ or a^{+} and generators close correct algebra)

Equal time bilocal fields

From OPE: bilocal packages the complete set of single trace primary operators

$$\begin{aligned}\sigma(t_1,\vec{x}_1,t_2,\vec{x}_2) &= \sum_{a=1}^{N}\phi^a(t_1,\vec{x}_1)\phi^a(t_2,\vec{x}_2)\\ &= \sum_{s=0}^{\infty}\sum_{d=0}^{\infty}c_{sd}\left(y^\mu\frac{\partial}{\partial x^\mu}\right)^d y_{\mu_1}\cdots y_{\mu_{2s}}j_{(2s)}^{\mu_1\cdots\mu_{2s}}(x)\end{aligned}$$

where $x^\mu = \frac{1}{2}(x_1^\mu + x_2^\mu)$ and $y^\mu = \frac{1}{2}(x_1^\mu - x_2^\mu)$.

Equal x^+ bilocal $\sigma(x^+,x_1^-,x_1,x_2^-,x_2)=\sum_a\phi^a(x^+,x_1^-,x_1)\phi^a(x^+,x_2^-,x_2)$ has $y^+=0$

$\Rightarrow$ packages only currents with $-$ and x polarizations.

Using known transformation rule of scalar field and the OPE, we verify symmetries are implemented as in the reduced theory obtained by eliminating $+$ polarizations.

The equal x^+ bilocal theory provides reduction of CFT to independent fields.

RdMK, JHEP **08** (2023), 056 2307.05032

Conformal symmetry

The equal x^+ bilocal theory provides reduction of CFT to independent fields.
Obtain a representation of symmetry on $j^{xx\cdots x}$ and $j^{-x\cdots x}$.

RdMK, JHEP **08** (2023), 056 2307.05032

To work out the so(2,3) AdS isometry (= conformal) generators after reducing to physical degrees of freedom we need to:

• Fix a gauge and solve the associated gauge constraint. Isometries are generated using the Killing vectors as usual.

• Since conformal transformations move out of lightcone gauge, each conformal transformation must be supplemented with a compensating gauge transformation, that restores the gauge.

• Reduce to independent degrees of freedom by solving the symmetric and traceless constraints.

Result is a set of transformation defined on $A^{XX\cdots X}$ and $A^{ZX\cdots X}$ fields.

Metsaev **NPB** 563 (1999) 295 hep-th/9906217

Many fields on AdS_4:

Co-ordinates: $X^+ \equiv X^2 + X^0 \qquad X^- \equiv X^2 - X^0, \qquad X \equiv X^1 \qquad Z$

Metric: $ds^2 = \frac{dX^+ dX^- + dX^2 + dZ^2}{Z^2}$

Fields: $A^{XX\cdots X}(X^+, X^-, X, Z)$, $A^{ZX\cdots X}(X^+, X^-, X, Z)$, $\Phi(X^+, X^-, X, Z)$

One field on $\mathrm{AdS}_4 \times \mathrm{S}^1$:

Co-ordinates: $X^+ \equiv X^2 + X^0 \qquad X^- \equiv X^2 - X^0, \qquad X \equiv X^1 \qquad Z \qquad \theta$

Metric: $ds^2 = \frac{dX^+ dX^- + dX^2 + dZ^2}{Z^2}$

Field: $\Phi(X^+, X^-, X, Z, \theta) = \sum_{s=0}^{\infty} \left(\cos(2s\theta) \frac{A^{XX\cdots XX}}{Z} + \sin(2s\theta) \frac{A^{XX\cdots XZ}}{Z} \right)$

Summary: Higher Spin Gravity

Equation of motion for physical d.o.f.:

$$\left(\frac{\partial}{\partial X^+}\frac{\partial}{\partial X^-}+\frac{\partial^2}{\partial X^2}+\frac{\partial^2}{\partial Z^2}\right)\frac{A^{XXZ\cdots ZX}}{Z}=0$$

Repackaged the complete set of physical and independent fields into a single field, which is a function of 5 co-ordinates:

$$\Phi(X^+,X^-,X,Z,\theta)=\sum_{s=0}^{\infty}\left(\cos(2s\theta)\frac{A^{XX\cdots XX}}{Z}+\sin(2s\theta)\frac{A^{XX\cdots XZ}}{Z}\right)$$

Action of conformal symmetry on $\frac{A^{XX\cdots X}}{Z}$, $\frac{A^{ZX\cdots X}}{Z}$ is known: for $L^A\in so(2,3)$

$$L^A_{\oplus}\Phi(X^+,X^-,X,Z,\theta)=\sum_{s=0}^{\infty}\left(\cos(2s\theta)L^A_{2s}\frac{A^{XX\cdots XX}}{Z}+\sin(2s\theta)L^A_{2s}\frac{A^{XX\cdots XZ}}{Z}\right)$$

Summary: Results for AdS$_d$

Equation of motion for physical d.o.f.:

$$\left(\frac{\partial}{\partial X^+}\frac{\partial}{\partial X^-}+\frac{\partial}{\partial \vec{X}}\cdot\frac{\partial}{\partial \vec{X}}+\frac{\partial^2}{\partial Z^2}+\frac{\mathcal{M}}{Z^2}\right)\frac{A^{XXZ\cdots ZX}}{Z}=0$$

Repackaged the complete set of physical and independent fields into a single field, which is a function of 5 co-ordinates:

$$\Phi(X^+,X^-,\vec{X},Z,\vec{\theta})=\sum_{\vec{s}} h(\vec{\theta},\vec{s})\frac{A^{XX\cdots XX}}{Z^{\frac{d-1}{2}}}$$

Action of conformal symmetry on $\frac{A^{XX\cdots X}}{Z}$, $\frac{A^{ZX\cdots X}}{Z}$ is known: for $L^A\in so(2,3)$

$$L^A_{\oplus}\Phi(X^+,X^-,X,Z,\theta)=\sum_{\vec{s}} h(\vec{\theta},\vec{s})L^A_{2s}\frac{A^{XX\cdots XX}}{Z}$$

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Bilocal Holography

$$\sigma(x^+, x_1^-, x_1, x_2^-, x_2) \quad = \quad \sigma_0(x^+, x_1^-, x_1, x_2^-, x_2) + \frac{1}{\sqrt{N}}\eta(x^+, x_1^-, x_1, x_2^-, x_2)$$

$\sigma_0(x^+, x_1^-, x_1, x_2^-, x_2)$ is the large N two point function.

$$\eta(x^+, x_1^-, x_1, x_2^-, x_2) \;\leftrightarrow\; \Phi(X^+, X^-, X, Z, \theta)$$

RdMK, Jevicki, Jin and Rodrigues, Phys. Rev. D **83** (2011) 025006.

The bilocal transforms in a tensor product. The complete collection of higher spin fields transform in a direct sum. The natural change of basis

$$V_{[\frac{1}{2},0]} \otimes V_{[\frac{1}{2},0]} \longrightarrow V_{[1,0]} \oplus \bigoplus_{s=2,4,\cdots} V_{[s+1,s]}$$

determines a map between CFT and bulk coordinates.

(Think of addition of angular momentum: $\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$.)

Change of Spacetime Co-ordinates

The bilocal transforms in $V_{\frac{1}{2},0} \otimes V_{\frac{1}{2},0}$ $\qquad (L^A \in \mathrm{so}(2,3))$

$$L^A_{\otimes}\sigma = \left(L^A\phi^a(x^+,x_1^-,x_1)\phi^a(x^+,x_2^-,x_2) + \phi^a(x^+,x_1^-,x_1)L^A\phi^a(x^+,x_2^-,x_2)\right)$$

$$V_{\frac{1}{2},0} \otimes V_{\frac{1}{2},0} = V_{1,0} \oplus \bigoplus_{s=2,4,6,\cdots} V_{s+1,s}$$

The complete collection of higher spin fields fill out the reducible representation $V_{1,0} \oplus \bigoplus_{s=2,4,6,\cdots} V_{s+1,s}$

$$L^A_{\oplus}\Phi(X^+,X^-,X,Z,\theta) = \sum_{s=0}^{\infty}\left(\cos(2s\theta)L^A_{2s}\frac{A^{XX\cdots XX}}{Z} + \sin(2s\theta)L^A_{2s}\frac{A^{XX\cdots XZ}}{Z}\right)$$

We want to change from the natural representation ($L^A_{\otimes}$) of the CFT to the representation that is natural for the bulk gravity ($L^A_{\oplus}$).

Change of Spacetime Coordinates

Bilocal field $\eta(x^+, x_1^-, x_1, x_2^-, x_2)$. 5 coordinates in CFT: $x^+, x_1^-, x_1, x_2^-, x_2$.

Higher spin gravity field $\Phi(X^+, X^-, X, Z, \theta)$. 5 coordinates in gravity: X^+, X^-, X, Z, θ.

Symmetry: $X^- \to X^- + a$ in gravity and $x^- \to x^- + b$ in CFT motivates the Fourier transform:

$$\eta(x^+, p_1^+, x_1, p_2^+, x_2) \;=\; \int \frac{dx_1^-}{2\pi} \int \frac{dx_2^-}{2\pi}\, \eta(x^+, x_1^-, x_1, x_2^-, x_2)\, e^{-ip_1^+ x_1^- - ip_2^+ x_2^-}$$

5 coordinates in CFT: $x^+, p_1^+, x_1, p_2^+, x_2$.

$$\Phi(X^+, P^+, X, Z, \theta) \;=\; \int \frac{dX^-}{2\pi}\, \Phi(X^+, X^-, X, Z, \theta)\, e^{-iP^+ X^-}$$

5 coordinates in gravity: X^+, P^+, X, Z, θ.

Change of Spacetime Coordinates

Bilocal field $\eta(x^+, p_1^+, x_1, p_2^+, x_2)$. 5 coordinates in CFT: $x^+, p_1^+, x_1, p_2^+, x_2$.
Higher spin gravity field $\Phi(X^+, P^+, X, Z, \theta)$. 5 coordinates in gravity: X^+, P^+, X, Z, θ.

$$x_1 = X + Z\tan\left(\frac{\theta}{2}\right) \qquad x_2 = X - Z\cot\left(\frac{\theta}{2}\right) \qquad x^+ = X^+$$

$$p_1^+ = P^+\cos^2\left(\frac{\theta}{2}\right) \qquad p_2^+ = P^+\sin^2\left(\frac{\theta}{2}\right)$$

$$X = \frac{p_1^+ x_1 + p_2^+ x_2}{p_1^+ + p_2^+} \qquad Z = \frac{\sqrt{p_1^+ p_2^+}}{p_1^+ + p_2^+}|x_1 - x_2|$$

$$P^+ = p_1^+ + p_2^+ \qquad \theta = 2\tan^{-1}\left(\sqrt{\frac{p_2^+}{p_1^+}}\right)$$

$$L_{\oplus}^A \Phi = 2\pi P^+ \sin\theta \; L_{\otimes}^A \eta$$

RdMK, Jevicki, Jin and Rodrigues, Phys. Rev. D **83** (2011) 025006.
RdMK, G. Kemp and J. Van Zyl, JHEP **04** (2024), 079, arXiv:2403.07606

Bulk locality

Consider an operator localized in the bulk of AdS at the point $X^M = (0, z_p)$.

$$P^\mu = \partial^\mu \qquad D = x\cdot\partial + z\partial_z + \frac{d-1}{2} \rightarrow z_p\partial_z + \frac{d-1}{2}$$

$$J^{\mu\nu} = x^\mu\partial^\nu - x^\nu\partial^\mu \rightarrow 0$$

$$K^\mu = -\frac{1}{2}(x\cdot x + z^2)\partial^\mu + x_\nu J^{\mu\nu} \rightarrow -\frac{1}{2}z_p^2\partial^\mu$$

The point is fixed by the isotropy group $\mathcal{G}$ generated by

$$\left\{ J^{\mu\nu}\,,\ \frac{z_p^2}{2}P^\mu + K^\mu \right\}$$

Isotropy group $\mathcal{G} = SO(1,d)$. The $d+1$ dimensional coset $SO(2,d)/\mathcal{G}$ is AdS$_{d+1}$.

Write the condition that a field is invariant under isotropy group. Fourier transform w.r.t. x^- and use generators of collective description.

Summary: Bilocal Holography

$$\begin{aligned}\sigma(x^+, x_1^-, x_1, x_2^-, x_2) &= \sum_{a=1}^{N} \phi^a(x^+, x_1^-, x_1)\phi^a(x^+, x_2^-, x_2) \\ &= \sigma_0(x^+, x_1^-, x_1, x_2^-, x_2) + \frac{1}{\sqrt{N}}\eta(x^+, x_1^-, x_1, x_2^-, x_2)\end{aligned}$$

$$X = \frac{p_1^+ x_1 + p_2^+ x_2}{p_1^+ + p_2^+} \qquad Z = \frac{\sqrt{p_1^+ p_2^+}}{p_1^+ + p_2^+}|x_1 - x_2| \qquad X^+ = x^+$$

$$P^+ = p_1^+ + p_2^+ \qquad \theta = 2\tan^{-1}\left(\sqrt{\frac{p_2^+}{p_1^+}}\right)$$

The collective field $\eta(x^+, x_1^-, x_1, x_2^-, x_2)$ is defined on $\mathrm{AdS}_4 \times \mathrm{S}^1$. The gravity fields are coefficients of a harmonic expansion on S^1. Note non-trivial field redfinition!

$$\begin{aligned}\Phi &= \sum_{s=0}^{\infty}\left(\cos(2s\theta)\frac{A^{XX\cdots XX}(X^+, P^+, X, Z)}{Z} + \sin(2s\theta)\frac{A^{XX\cdots XZ}(X^+, P^+, X, Z)}{Z}\right) \\ &= 2\pi P^+ \sin\theta \; \eta(X^+, P^+\cos^2\frac{\theta}{2}, X + Z\tan\frac{\theta}{2}, P^+\sin^2\frac{\theta}{2}, X - Z\cot\frac{\theta}{2})\end{aligned}$$

Two time bilocals

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AdS maps and diagrams of bi-local holography

Robert de Mello Koch,[a,b] Antal Jevicki,[c] Kenta Suzuki[c,d] and Junggi Yoon[e]

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A derivation of AdS/CFT for vector models

Ofer Aharony, Shai M. Chester and Erez Y. Urbach

These use unequal time bilocals $\sigma(x_1^\mu, x_2^\mu)$ - no reduction to physical / independent degrees of freedom. Bulk fields are extracted using conformal partial waves.

Further Examples

Following a completely parallel analysis, we can construct the holographic mapping for the collective field σ_k relevant for the description of matrix models or Yang-Mills theories.

The k local collective field is defined on the space

$$\mathrm{AdS}_{d+1}\times \mathrm{S}^{k-1}\times \mathrm{S}^{d-3}\times \mathrm{S}^{(k-2)(d-2)}$$

The coefficients of the harmonic expansion of σ_k on $\mathrm{S}^{k-1}\times \mathrm{S}^{d-3}\times \mathrm{S}^{(k-2)(d-2)}$ are the bulk fields defined on AdS_{d+1}.

$$\Phi(X^+,P^+,X^a,Z,\{\alpha_i\}) = Z^{\frac{3-d}{2}}\,(P^+Z)^{\frac{k}{2}(d-2)-1}\,\Big(\prod_{i=1}^{k} p_i^+\Big)^{\frac{4-d}{2}}\eta_k(x^+,p_1^+,x_1^a,\cdots,p_k^+,x_k^a)$$

The basis functions of the harmonic function diagonalize the AdS mass operator consistent with the GKPW rule.

$$Z \;=\; \frac{\sqrt{\sum_{a=1}^{d-2}\sum_{i=1}^{k} p_i^+ (v_i^a)^2}}{\left(\sum_{l=1}^{k} p_l^+\right)^{\frac{3}{2}}} \qquad v_i^a \;=\; \sum_{l=1}^{k} p_l^+(x_i^a - x_l^a)$$

Outline

What is a collective field?

Why use collective fields?

Matching degrees of freedom (AdS_4/CFT_3)

Holographic map between collective field theory and gravity (AdS_4/CFT_3)

Redundancy and holography (AdS_4/CFT_3)

Testing the holographic map (AdS_4/CFT_3)

Conclusions.

Redundancy in Collective Field Theory

In both the path integral (unequal time collective fields) and the canonical quantization (equal time collective fields) approaches we treat all degrees of freedom in the collective field as independent.

$$\langle \cdots \rangle = \int d\sigma(x_1^\mu, x_2^\mu) e^{iS} \cdots$$

$$[\Pi(t,\vec{x}_1,\vec{x}_2), \sigma(t,\vec{y}_1,\vec{y}_2)] = -i\delta(\vec{x}_1-\vec{y}_1)\delta(\vec{x}_2-\vec{y}_2) \qquad \Pi(t,\vec{x}_1,\vec{x}_2) = \frac{1}{i}\frac{\delta}{\delta\sigma(t,\vec{x}_1,\vec{x}_2)}$$

This is a very redundant description compared to the original description

$$\langle \cdots \rangle = \int d\phi^a(x^\mu) e^{iS} \cdots$$

$$[\pi^a(t,\vec{x}), \phi^b(t,\vec{y})] = -i\delta^{ab}\delta(\vec{x}-\vec{y}) \qquad \pi^a(t,\vec{x}) = \frac{1}{i}\frac{\delta}{\delta\phi^a(t,\vec{x})}$$

One independent degree of freedom at each point $(\vec{x}_1,\vec{x}_2)$ versus N independent degrees of freedom at each $\vec{x}$.

Can we get some insight into this redundancy?

Operators deep in the bulk

Which collective fields map to operators localized deep in the bulk of AdS_4?

$$Z = \frac{\sqrt{p_1^+ p_2^+}}{p_1^+ + p_2^+}|x_1 - x_2|$$

p_1^+ and p_2^+ are both positive $\Rightarrow$ the ratio $0 < \frac{\sqrt{p_1^+ p_2^+}}{p_1^+ + p_2^+} < 1$.

To obtain a large value for Z we must consider a large separation between the two fields in the bilocal, i.e. $x_1 - x_2$ must be large.

Location (in the bulk) of single trace primaries

Where do single trace primaries map to in the AdS_4 bulk? ($Z = \frac{\sqrt{p_1^+ p_2^+}|x_1 - x_2|}{p_1^+ + p_2^+}$)

Scalar primary $= \phi^a(x^+, x^-, x)\phi^a(x^+, x^-, x)$ i.e. located on boundary $Z = 0$.

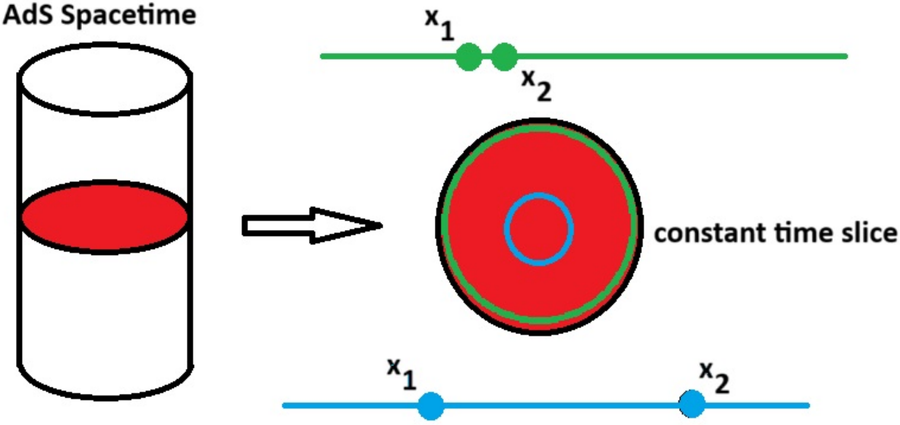
Conserved currents are

$$
\begin{aligned}
J_s(x^+, x^-, x, \alpha) &= J_{\mu_1\mu_2\cdots\mu_s}(x^+, x^-, x)\alpha^{\mu_1}\alpha^{\mu_2}\cdots\alpha^{\mu_s} \\
&= \sum_{k=0}^{s} \frac{(-1)^k \ :(\alpha\cdot\partial)^{s-k}\phi^a(x^+, x^-, x)\ (\alpha\cdot\partial)^k\phi^a(x^+, x^-, x):}{k!(s-k)!\Gamma(k+\frac{1}{2})\Gamma(s-k+\frac{1}{2})} \\
&= \sum_{k=0}^{s} \frac{(-1)^k\,(\alpha\cdot\partial_1)^{s-k}\ (\alpha\cdot\partial_2)^k}{k!(s-k)!\Gamma(k+\frac{1}{2})\Gamma(s-k+\frac{1}{2})}\eta(x^+, x_1^-, x_1, x_2^-, x_2)\Big|_{x_1=x_2=x, x_1^-=x_2^-=x^-}
\end{aligned}
$$

To construct spinning currents, separate x_1 and x_2 by a small amount ϵ, evaluate the derivatives and then send $x_2 \to x_1$. Take $|x_1 - x_2| < \epsilon$ where ϵ can be arbitrarily small, $\Rightarrow Z < \epsilon$.

SINGLE TRACE PRIMARIES MAP TO NEIGHBOURHOOD OF BOUNDARY $Z = 0$.

Operators deep in the bulk



$$\sigma(x^+, p_1^+, x_1, p_2^+, x_2) \;=\; \phi^a(x^+, p_1^+, x_1)\phi^a(x^+, p_2^+, x_2)$$

Remarks related to the OPE

Single trace primaries are supported in arbitrarily small neighbourhood of the boundary.

By separating x_1 and x_2 to be arbitrarily distant, the bilocal $\eta(x^+, x_1^-, x_1, x_2^-, x_2)$ is located arbitrarily deep in the bulk ($Z = \frac{\sqrt{p_1^+ p_2^+}|x_1 - x_2|}{p_1^+ + p_2^+}$).

The OPE states

$$\begin{aligned}\eta(x^+, x_1^-, x_1, x_2^-, x_2) &= :\phi^a(x^+, x_1^-, x_1)\phi^a(x^+, x_2^-, x_2): \\ &= \sum_s \lambda_{\phi\phi J} C(x_1^- - x_2^-, x_1 - x_2)_{\mu_1 \cdots \mu_{2s}} J^{\mu_1 \cdots \mu_{2s}}(x^+, x_1^-, x_1)\end{aligned}$$

This replaces an operator located deep in the bulk of AdS_4 by a sum of operators located in an arbitrarily small neighbourhood of the boundary of AdS_4.

Holography from Redundancy

In both the path integral (unequal time collective fields) and the canonical quantization (equal time collective fields) approaches we treat all degrees of freedom in the collective field as independent.

This is a very redundant description of the original conformal field theory. The complete set of single trace primary operators live in the neighbourhood of the boundary of AdS_{d+1} and any collective field (which may correspond to a degree of freedom deep in the bulk) can be expressed in as a sum over the single trace primaries and their descendants.
$\Rightarrow$ **a complete set of degrees of freedom reside on the boundary of spacetime.**

Thus, the redundancy present in the collective field theory description naturally makes this description a holographic description i.e. a gravitational description.

RdMK, JHEP **10** (2023), 151, 2309.11116

This redundancy matches that in quantum gravity, arising from the gravitational gauge symmetry. Chowdhury, Godet, Papadoulaki, Raju, JHEP **03** (2022), 019, 2107.14802.

Outline

Bulk Reconstruction

Does the proposed bulk field $\Phi(X^+, P^+, X, Z, \theta)$ obey the correct bulk equation of motion with the correct boundary condition?

The bulk equation of motion in the fully gauge fixed gravity is

$$\left(\frac{\partial}{\partial X^+}\frac{\partial}{\partial X^-} + \frac{\partial}{\partial \vec{X}} \cdot \frac{\partial}{\partial \vec{X}} + \frac{\partial^2}{\partial Z^2} + \frac{\mathcal{M}}{Z^2}\right) \frac{A^{XXZ\cdots ZX}}{Z} = 0$$

The AdS mass operator $\mathcal{M}$ when translated back to CFT_d is a function of the quadratic conformal Casimir of CFT_d.

The bulk equation of motion becomes the statement that we should extract a particular primary operator from the collective field.

Bulk Reconstruction

Does the proposed bulk field $\Phi(X^+, P^+, X, Z, \theta)$ obey the correct bulk equation of motion with the correct boundary condition?

The identity:

$$(p_1^+ + p_2^+)^s \cos\left(2s\tan^{-1}\sqrt{\frac{p_2^+}{p_1^+}}\right) = \mathcal{N}\sum_{k=0}^{s}\frac{(-1)^k (p_1^+)^{s-k}(p_2^+)^k}{\Gamma\left(s-k+\frac{1}{2}\right)\Gamma\left(k+\frac{1}{2}\right)k!(s-k)!}$$

implies that ($\mathcal{N} = \Gamma\left(\frac{1}{2}\right) s!\Gamma\left(s+\frac{1}{2}\right)$; recall $\Phi \sim \cos(2s\theta)\frac{A^{X\cdots X}}{Z} + \sin(2s\theta)\frac{A^{X\cdots XZ}}{Z}$)

$$\frac{\partial^s}{\partial X^{-s}}\Phi_s(X^+; X^-, X, 0) = 16\pi\mathcal{N}\sum_{k=0}^{s}\frac{(-1)^k \partial_-^{s-k}\phi^a(X^+, X^-, X)\partial_-^k\phi^a(X^+, X^-, X)}{\Gamma\left(s-k+\frac{1}{2}\right)\Gamma\left(k+\frac{1}{2}\right)k!(s-k)!}$$

where $\Phi_s \equiv \frac{A^{XX\cdots XX}}{Z}$. (dMK, Jevicki, Rodrigues, Yoon, J. Phys. A **48** (2015) 105403, 1408.4800)
Recall that

$$J_{\mu_1\mu_2\cdots\mu_{2s}}(t,\vec{x})\alpha^{\mu_1}\alpha^{\mu_2}\cdots\alpha^{\mu_{2s}} = \sum_{a=1}^{N}\sum_{k=0}^{2s}\frac{(-1)^k\ :(\alpha\cdot\partial)^{2s-k}\phi^a\ (\alpha\cdot\partial)^k\phi^a:}{k!(2s-k)!\Gamma(k+\frac{1}{2})\Gamma(2s-k+\frac{1}{2})}$$

Bulk Reconstruction

To motivate why we consider the identity:

$$(p_1^+ + p_2^+)^s \cos\left(2s\tan^{-1}\sqrt{\frac{p_2^+}{p_1^+}}\right) = \mathcal{N}\sum_{k=0}^{s}\frac{(-1)^k (p_1^+)^{s-k}(p_2^+)^k}{\Gamma\left(s-k+\frac{1}{2}\right)\Gamma\left(k+\frac{1}{2}\right)k!(s-k)!}$$

recall that the map includes the entries:

$$\theta \;=\; 2\tan^{-1}\left(\sqrt{\frac{p_2^+}{p_1^+}}\right)$$

$$\begin{aligned}\Phi \;&=\; \sum_{s=0}^{\infty}\left(\cos(2s\theta)\frac{A^{XX\cdots XX}(X^+,P^+,X,Z)}{Z} + \sin(2s\theta)\frac{A^{XX\cdots XZ}(X^+,P^+,X,Z)}{Z}\right)\\ &=\; 2\pi P^+ \sin\theta\ \eta(X^+, P^+\cos^2\frac{\theta}{2}, X + Z\tan\frac{\theta}{2}, P^+\sin^2\frac{\theta}{2}, X - Z\cot\frac{\theta}{2})\end{aligned}$$

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where $\Phi_s \equiv \frac{A^{XX\cdots XX}}{Z}$. (dMK, Jevicki, Rodrigues, Yoon, J. Phys. A **48** (2015) 105403, 1408.4800)

Recall that

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Consistency with GKPW dictionary

GKPW dictionary is formulated in de Donder gauge: $D^A A_{AA_2\cdots A_{2s}} = 0$. Residual gauge symmetry can be used to make $A_{A_1A_2\cdots A_{2s}}$ traceless.

From e.o.m. near $Z = 0$ we have (M is a boundary index - does not take Z values)

$$A_{M_1\cdots M_{2s}} \sim Z^{2-2s} A^{\rm non-norm}_{M_1\cdots M_{2s}}(X^+, X^-, X) + Z^{1+2s} A^{\rm norm}_{M_1\cdots M_{2s}}(X^+, X^-, X)$$

$$\begin{aligned} H_{M_1\cdots M_{2s-k}Z\cdots Z} \quad \sim \quad & Z^{2-2s-k} A^{\rm non-norm}_{M_1\cdots M_{2s-k}Z\cdots Z}(X^+, X^-, X) \\ & +Z^{1+2s+k} A^{\rm norm}_{M_1\cdots M_{2s-k}Z\cdots Z}(X^+, X^-, X) \end{aligned}$$

GKPW says:

$$j_{M_1,M_2\cdots M_{2s}} \propto A^{\rm norm}_{M_1,M_2\cdots M_{2s}}$$

CFT operator is related to component of boundary field falling off as Z^{1+2s} and the relation is local!

Consistency with GKPW dictionary

Bilocal holography is formulated in light cone gauge and not de Donder gauge.

Transform to lightcone gauge: $A'_{A_1 A_2 \cdots A_{2s}} = A_{A_1 A_2 \cdots A_{2s}} - D_{(A_1} \Lambda_{A_2 \cdots A_{2s})}$

Requiring $A'_{+A_2 \cdots A_{2s}} = 0$ fixes the gauge parameter $\Lambda_{A_1 \cdots A_{2s-1}}$.

Example: Spin $2s$ After the gauge transformation, GKPW says

$$A'_{XXX \cdots X} = -A'_{ZZX \cdots X} \propto Z \partial_-^{-2s} j_{--\cdots--}$$

$$\Rightarrow \partial_-^{2s} \frac{A'_{XXX \cdots X}}{Z} = -\partial_-^{2s} \frac{A'_{ZZX \cdots X}}{Z} \propto j_{--\cdots--}$$

so bilocal holography gives a correct bulk reconstruction.

Mintun and Polchinski, arXiv:1411.3151

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Conclusions

Discussion and Future Directions

Using a collective field theory treatment of CFT we have constructed a higher dimensional gravitational theory.

The resulting holographic map exhibits enough features expected of the quantum gravity dual to the original conformal field theory, that it is a convincing example of constructive holography.

Much of what has been achieved used mainly input from representation theory of conformal group. This can be extended to use the impressive results for planar spectrum of anomalous dimensions. To appear soon.

$1/N$ corrections: match interactions generated by collective field theory Jacobian with the non-linear interactions of gravity.

More general gauge theory/gravity dualities...

Thanks for your attention!