Tracy-Widom distribution in four-dimensional superconformal Yang-Mills theories

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International Workshop on Exact Methods in Quantum Field Theory and String Theory

November 1, 2024

Outline and summary

Ultimate goal is to solve 4d superconformal $\mathcal{N}=2$ and $\mathcal{N}=4$ planar Yang–Mills theories for arbitrary 't Hooft coupling $\lambda = g^2_{\rm YM} N_c$

Existing techniques (localization, integrability, holography) allows us to realize this program for:

- ✔ V.e.v. of half-BPS circular Wilson loop in $\mathcal{N}=4$ SYM
- ✔ Correlation function of infinitely heavy half-BPS operators (= octagon)
- ✔ Flux tube correlators (cusp anom. dim., scattering amplitudes)
- ✔ Free energy and correlation functions in $\mathcal{N}=2$ SYM

A remarkable feature of these observables is that they can be expressed as determinants of certain semi-infinite matrices

$$e^{\mathcal{F}(g)} = \det_{1\le n,m<\infty}\Big(\delta_{nm} - K_{nm}(g)\Big)\,, \qquad\qquad g = \frac{\sqrt{\lambda}}{4\pi}$$

We shall compute $\mathcal{F}(g)$ for arbitrary g

Weak and strong coupling expansion

Simple example: circular Wilson loop in planar $\mathcal{N} = 4$ SYM

$$W = \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda})$$

Weak coupling expansion

$$W \overset{\lambda \ll 1}{=} 1 + \frac{\lambda}{8} + \frac{\lambda^2}{192} + \frac{\lambda^3}{9216} + \frac{\lambda^4}{737280} + \dots$$

Strong coupling expansion

$$W \overset{\lambda \gg 1}{=} e^{\sqrt{\lambda} - \frac{3}{2}\log\sqrt{\lambda} - \frac{1}{2}\log\left(\frac{\pi}{2}\right) - \frac{3}{8\sqrt{\lambda}} - \frac{3}{16\lambda} + \dots} + O(e^{-\sqrt{\lambda}})$$

Semiclassical asymptotics of observables in AdS/CFT

$$\mathcal{F} = -\sqrt{\lambda} A_0 - A_1 \log(\sqrt{\lambda}) - B - \sum_{n \geq 1} \frac{A_{n+1}}{\lambda^{n/2}} + O(e^{-c\sqrt{\lambda}})$$

- ✔ Expansion coefficients grow factorially $A_n \sim n!$
- ✔ Needs to account for nonperturbative (exponentially small) corrections

Tracy-Widom distribution

Describes statistics of the spacing of the eigenvalues of $N \times N$ hermitian matrices for $N \to \infty$

Gaussian Unitary Ensemble

$$Z_{\rm GUE} = \int d^{N\times N} a\, e^{-\frac{1}{2}\,{\rm tr}\, a^2} = \int_{-\infty}^{\infty} d\mu_1 \ldots d\mu_N \prod_{i\neq j} (\mu_i - \mu_j)^2 e^{-\frac{1}{2}\sum_i \mu_i^2}$$

Laguerre ensemble (Wishart matrix theory)

$$Z_{\rm Laguerre} = \int_0^{\infty} d\mu_1 \ldots d\mu_N \prod_{i\neq j} (\mu_i - \mu_j)^2 \prod_{i=1}^{N} \mu_i^{\ell}\, e^{-\mu_i}$$

where $\ell > -1$ and eigenvalues are located on semi-axis $[0, \infty)$.

The probability density for eigenvalues

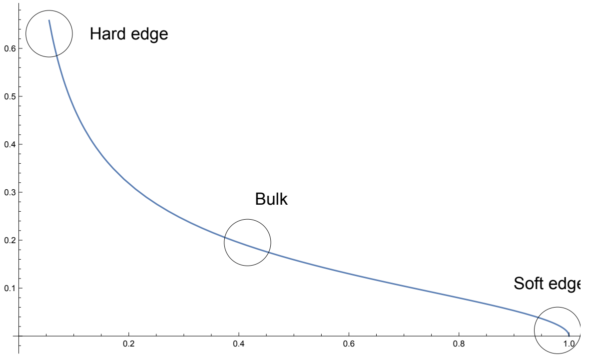
$$R_n(x_1, \ldots, x_n) = \left\langle \prod_{i=1}^{n} \delta(\mu_i - x_i) \right\rangle = \det K_N(x_i, x_j)\Big|_{i,j=1,\ldots,n}$$

$$K_N(x,y) = \sum_{k=0}^{N-1} \phi_k(x)\phi_k(y)$$

where $\phi_k(x)$ are orthonormal functions $x^k e^{-x^2/2} + \ldots$ (GUE) and $x^k x^{\ell/2} e^{-x/2} + \ldots$ (Laguerre)

Tracy-Widom distribution II

The distribution of the eigenvalues in the Laguerre ensemble in the limit $N \to \infty$



$$R_1(4Nx) \sim \frac{1}{2\pi}\sqrt{\frac{1-x}{x}}$$

Scaling behaviour of $K_N(x,y)$ around $x=0$ (hard edge), $x=1$ (soft edge) and $0<x<1$ (bulk)

bulk : $$\frac{\sin\pi(x-y)}{\pi(x-y)}$$

soft edge : $$\frac{\mathrm{Ai}(x)\mathrm{Ai}'(y)-\mathrm{Ai}(x)\mathrm{Ai}'(y)}{x-y}$$

hard edge : $$\frac{J_\ell(\sqrt{x})\sqrt{y}J_\ell'(\sqrt{y})-\sqrt{x}J_\ell'(\sqrt{x})J_\ell(\sqrt{y})}{2(x-y)}$$

The probability that there are no eigenvalues on the interval $[0,s]$

$$E(0;s)=\det(1-K)_{[0,s]}=1+\sum_{n\geq 1}\frac{(-1)^n}{n!}\int_0^s dx_1\ldots dx_n \det\|K(x_i,x_j)\|_{1\leq i,j\leq n}$$

Fredholm determinant of the integral operator: *Sinc* (bulk), *Airy* (soft edge) and *Bessel* (hard edge)

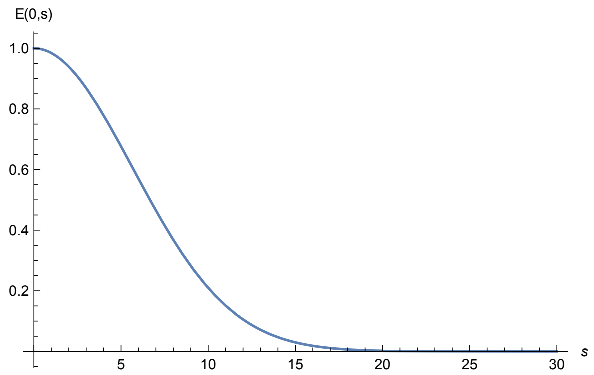
Bessel kernel

Tracy-Widom distribution close to the hard edge

$$E(0,s) = \det(1 - K_{\text{Bessel}})_{[0,s]} = \exp\left(-\frac{1}{4}\int_0^s dx \log(s/x)\, Q^2(x)\right)$$

$Q(s)$ satisfies Painlevé V differential equation

Dependence of the probability $E(0,s)$ on the interval length s



Asymptotics of $E(0,s)$ at small and large s

$$E(0,s) \overset{s\ll 1}{=} 1 - \frac{(s/4)^{\ell+1}}{\Gamma^2(\ell+2)} + \dots$$

$$E(0,s) \overset{s\gg 1}{=} \exp\left(-\frac{s}{4} - \frac{\ell^2}{4}\log s + \frac{\ell}{8}s^{-1/2} + \dots\right)$$

Remarkably similar to weak/strong coupling expansion in gauge theory for $s \sim \sqrt{\lambda}$

Bessel kernel at finite temperature

$$K_\ell(x,y) = \sum_{n\geq 1} \phi_n(x)\phi_n(y)\chi\left(\frac{y}{2g}\right)$$

$$\phi_n(x) = (-1)^n\sqrt{2n+\ell-1}\,\frac{J_{2n+\ell-1}(\sqrt{x})}{\sqrt{x}}$$

- ✔ For $\chi(x) = \theta(1-x)$ defines the Tracy-Widom distribution $E(0,s)$ for $s = (2g)^2$
- ✔ Finite-temperature generalization: $\chi(x) = 1/(1+e^{\frac{x-\mu}{T}})$

Generalized Tracy-Widom distribution

$$e^{\mathcal{F}_\chi} = \det(1-K_\ell) = \det\Big(\delta_{nm} - K_{nm}(g)\Big)\Big|_{n,m\geq 1}$$

Determinant of a semi-infinite matrix

$$\int_0^\infty dy\, K_\ell(x,y)\,\phi_n(y) = K_{nm}\phi_m(x)$$

$$K_{nm} = \int_0^\infty dx\,\phi_n(x)\phi_m(x)\chi\left(\frac{x}{2g}\right)$$

$\chi(x)$ is the *symbol* of the Bessel operator

Free energy in $\mathcal{N}=2$ super Yang-Mills theory

- ✔ $\mathcal{N}=2$ supersymmetric Yang-Mills theory with gauge group $SU(N)$ coupled to matter multiplets

  in rank-2 symmetric $(N_S=1)$ and anti-symmetric $(N_A=1)$ representations

  The beta function vanishes $\beta_0 = 2N - N_S(N+2) - N_A(N-2) = 0$

- ✔ The partition function on sphere S^4 is given by a matrix integral [Pestun]

$$Z_{S^4} = e^{-F} = \int da\, e^{-\frac{8\pi^2 N}{\lambda}\,\mathrm{tr}\, a^2} |Z_{1-\mathrm{loop}}(a) Z_{\mathrm{inst}}(a)|^2$$

  Non-perturbative instanton contribution $Z_{\mathrm{inst}}(a)$ is exponentially small at large N

- ✔ Perturbative corrections $Z_{1-\mathrm{loop}}(a) = \exp(-S_{\mathrm{int}}(a))$ only come from one loop

$$S_{\mathrm{int}}(a) = \sum_{i,j}\Big[\log H(\mu_i+\mu_j) - \log H(\mu_i-\mu_j)\Big] \qquad (\mu_i \text{ are eigenvalues of } a)$$

$$= 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n+1}\zeta_{2n+1}\sum_{p=0}^{n}\binom{2n+2}{2p+1}\,\mathrm{tr}\, a^{2p+1}\ \mathrm{tr}\, a^{2(n-p)+1}$$

$$H(x) = \prod_{n=1}^{\infty}\left(1+\frac{x^2}{n^2}\right)^n e^{-\frac{x^2}{n}} = \exp\left(\sum_{n=1}^{\infty}\frac{(-1)^n}{n+1}\zeta_{2n+1}\, x^{2n+2}\right)$$

  Matrix model with double-trace interaction

$$e^{-F} = \left(\frac{8\pi^2}{\lambda}\right)^{-(N^2-1)/2} \int da\, \exp\left(-N\,\mathrm{tr}\, a^2 + \frac{1}{2}\sum_{k,n} C_{kn} O_{2k+1} O_{2n+1}\right)$$

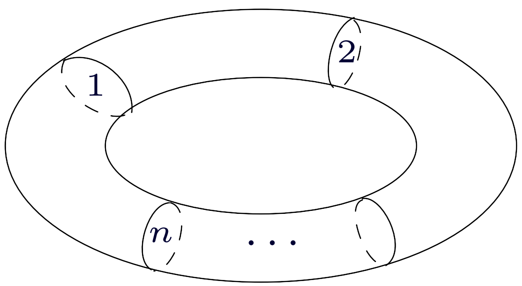
The interaction term is a sum over double traces $O_k = \mathrm{tr}\, a^k$ with the couplings

$$C_{kn} = 4\frac{(-1)^{k+n+1}}{k+n+1}\zeta_{2(k+n)+1}\binom{2(k+n+1)}{2k+1}\left(\frac{\lambda}{8\pi^2}\right)^{k+n+1}$$

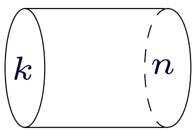
Large N expansion

$$F = N^2 F_0(\lambda) + F_1(\lambda) + F_2(\lambda)/N^2 + \dots$$

The interaction term does not contribute to F_0 – coincides with the free energy in $\mathcal{N} = 4$ SYM

$F_1 = \sum_{n\geq 1}$ 

$= -\sum_{n\geq 1} \frac{1}{2n}\,\mathrm{tr}[(QC)^n] = \frac{1}{2}\log\det(1-QC)$

Cylinders $Q_{kn} =$ 

$= \langle \mathrm{tr}\, a^{2k+1}\, \mathrm{tr}\, a^{2n+1}\rangle_{\mathrm{GUE}}$ are glued together with the weight C_{kn}

Relation to Bessel kernel

Explicit expressions for semi-infinite matrices

$$Q_{kn} = \frac{2\beta_k \beta_n}{k+n+1} + O(1/N^2)\,, \qquad \beta_n = \frac{2^n n \Gamma(n+\frac{3}{2})}{\sqrt{\pi}\Gamma(n+2)}$$

$$C_{kn} = 4\frac{(-1)^{k+n+1}}{k+n+1}\zeta_{2(k+n)+1}\binom{2(k+n+1)}{2k+1}\left(\frac{\lambda}{8\pi^2}\right)^{k+n+1}$$

The matrix (QC) is related to the Bessel kernel by a similarity transformation

[Beccaria,Billò,Galvagno,Hasan,Lerda]

$$\begin{aligned} K_{nm} &= (U^{-1}QCU)_{nm} \\ &= 2\,(-1)^{n+m}\sqrt{2n+1}\sqrt{2m+1}\int_0^\infty \frac{dt}{t} J_{2n+1}(t)\, J_{2m+1}(t)\chi\left(\frac{x}{2g}\right) \end{aligned}$$

Special form of the symbol

$$\chi(x) = -\frac{1}{\sinh^2(x/2)}\,, \qquad g = \frac{\sqrt{\lambda}}{4\pi}$$

The free energy coincides with the Tracy-Widom distribution at the hard edge for $\ell = 2$

$$F_1 = \frac{1}{2}\log\det(1 - QC) = \frac{1}{2}\,\mathrm{tr}\log(1 - \mathbf{K}_\chi)$$

Tracy-Widom distribution in super Yang-Mills theories

Different observables in SYM theories are given by the Tracy-Widom distribution $e^{\mathcal{F}_\chi}$

The symbol function χ depends on the observable:

- ✔ Circular Wilson loop

$$\chi_{\mathsf{W}}(x) = -\frac{(2\pi)^2}{x^2}$$

- ✔ Free energy of $\mathcal{N} = 2$ SYM

$$\chi_{\mathsf{free}}(x) = -\frac{1}{\sinh^2(x/2)}$$

- ✔ Four-point correlator

$$\chi_{\mathsf{oct}}(x|y,\xi) = \frac{\cosh y + \cosh \xi}{\cosh y + \cosh(\sqrt{x^2+\xi^2})}$$

- ✔ Flux tube

$$\chi_{\mathsf{flux}}(x) = -\frac{2}{e^x - 1}$$

The coupling constant $g = \sqrt{\lambda}/(4\pi)$ controls the width of the distribution $s \sim g^2$

How to derive the strong coupling expansion of the TW distribution?

Szegő-Akhiezer-Kac formula

- Asymptotic behaviour for sufficiently smooth symbol $\chi(z)$

$$\mathcal{F}_\chi = -gA_0 + B + O(1/g)$$ *SAK formula (1915-1966)*

$$A_0 = -2\widetilde{\psi}(0)\,, \qquad B = \frac{1}{2}\int_0^\infty dk\, k\big(\widetilde{\psi}(k)\big)^2\,,$$

$$\widetilde{\psi}(k) = \int_0^\infty \frac{dz}{\pi}\cos(kz)\log(1-\chi(z))$$

B diverges for $\chi(z)\sim 1-z^{2\beta}$ or $\widetilde{\psi}(k)\sim -\beta/k$ at large k *Fisher-Hartwig singularity*

The SAK formula for the Bessel kernel with Fisher-Hartwig singularity has not been derived yet

- Our conjecture [Belitsky,GK]

$$\mathcal{F}_\chi = -gA_0 + A_1\log g + B' + O(1/g)$$

$$A_1 = \frac{1}{2}\beta^2\,,$$

$$B' = \frac{1}{2}\int_0^\infty dk\,\left[k\big(\widetilde{\psi}(k)\big)^2 - \beta^2\frac{1-\mathrm{e}^{-k}}{k}\right] + \frac{\beta}{2}\log(2\pi) - \log G(1+\beta)\,,$$

Power suppressed $O(1/g)$ corrections are determined using the *method of differential equations*

Method of differential equations

A powerful method for computing correlators in integrable models [Its,Izergin,Korepin,Slavnov]

'Potential' = logarithmic derivative of the determinant

$$U(g) = -2g\partial_g \mathcal{F}_\chi(g)$$

Satisfies the system of *exact* integro-differential equations [Belitsky,GK]

$$g\partial_g U = -2\int_0^\infty dx\, Q^2(x)\, x\partial_x \chi\left(\frac{\sqrt{x}}{2g}\right),$$

$$(g\partial_g + 2x\partial_x)^2\, Q(x) + (x - g\partial_g U + U)\, Q(x) = 0$$

- ✔ For $\chi(x) = \theta(1-x)$ reduces to Painlevé V equation for $q(g) = Q(x=(2g)^2)$

$$(g\partial_g)^2\, q(g) + \left(4g^2 - g\partial_g U + U\right) q(g) = 0\,, \qquad g\partial_g U = [q(g)]^2$$

- ✔ For generic $\chi(x)$ exact solution is not known, WKB solution at large g

$$Q((2gz)^2) = \frac{a(z,g)\sin(2gz) + a(-z,g)\cos(2gz)}{\sqrt{2\pi g z(1-\chi(z))}}\,, \qquad a(z,g) = 1 + \sum_{k\ge 1}\frac{a_k(z)}{g^k}$$

Tracy-Widom distribution at strong coupling

- Strong coupling expansion:

$$\mathcal{F}_\chi(g) = \underbrace{-gA_0 + A_1 \log g + B}_{\text{SAK formula}} + f(g) + \Delta f(g)$$

- The ‘perturbative’ function $f(g)$ is given by an asymptotic series

$$f(g) = \sum_{k=1}^{\infty} \frac{A_{k+1}}{2k(k+1)} g^{-k}$$

- The expansion coefficients $A_k = A_k(\chi)$ depend on the symbol function (=choice of observable)

  Curious relation between two different observables

$$A_{k+1}(\chi_{\text{free}}) = (-1)^k A_{k+1}(\chi_{\text{oct}}) \qquad \mapsto \qquad f_{\text{free}}(g) = f_{\text{oct}}(-g)$$

- The expansion coefficients grow factorially $A_k \sim k!\, c^{-k}$

  The perturbative series $f(g)$ is plagued with Borel singularities

- Has to be supplemented with the nonperturbative, exponentially small corrections

$$\Delta f(g) \sim e^{-cg}$$

Nonperturbative corrections

We developed a systematic method to compute transseries for $\Delta f(g)$

$$\Delta f(g) = e^{-c_1 g} f_1(g) + e^{-c_2 g} f_2(g) + \dots\,, \qquad f_n(g) = \sum_k f_n^{(k)} g^{-k}$$

Sum over saddles in AdS/CFT

General form of the symbol function in SYM

$$1 - \chi(x) = b x^{2\beta} \prod_{n\geq 1} \frac{1 + \frac{x^2}{(2\pi x_n)^2}}{1 + \frac{x^2}{(2\pi y_n)^2}}$$

Has an infinite set of poles and zeros located at $x = -2i\pi x_n$ and $x = -2i\pi y_n$, e.g.

$$1 - \chi_{\text{oct}}(x|0,0) = \frac{\sinh^2(x/2)}{\cosh^2(x/2)} = \frac{x^2}{4} \prod_{n\geq 1} \left[\frac{1 + \frac{x^2}{(2\pi n)^2}}{1 + \frac{x^2}{(2\pi(n-\frac{1}{2}))^2}}\right]^2$$

Nonperturbative corrections

$$\Delta f(g) = \sum_{n\geq 1} \left(g^{a} e^{-8\pi g x_1}\right)^n \left[f_n^{(0)} + \sum_{k=1}^{\infty} f_n^{(k)} g^{-k}\right]$$

The parameter x_1 is a solution to $\chi(2i\pi x_1) = 1$ closest to the origin, with degeneracy $a + 1 = 1, 2$

Octagon

Perturbative part

$$f_{\text{oct}}(g) = \frac{3}{8}\log(g'/g) + \frac{15\zeta_3}{32(4\pi g')^3} + \frac{945\zeta_5}{256(4\pi g')^5} - \frac{765\zeta_3^2}{64(4\pi g')^6} + \dots$$

Nonperturbative part

$$\Delta f_{\text{oct}}(g) = \frac{i\pi g'}{4} e^{-8\pi g}\left[1 - \frac{\frac{7}{4}}{(4\pi g')} - \frac{\frac{63}{32}}{(4\pi g')^2} + \dots\right] + \frac{(\pi g')^2}{32} e^{-16\pi g}\left[1 + \frac{\frac{81i}{4} - \frac{7}{2}}{4\pi g'} + \frac{-\frac{1431i}{32} - \frac{3}{4}}{(4\pi g')^2} + \dots\right] + O(e^{-24\pi g})$$

Finite renormalization of the coupling $g' = g + \log(2)/\pi$

The expansion coefficients grow factorially at large orders

$$f_{\text{oct}}(g) = \sum_{n\geq 1} \frac{\alpha_n}{(4\pi g')^n}\,, \qquad \alpha_n \sim \Gamma(n+1)$$

The same is true for the coefficient functions in $\Delta f_{\text{oct}}(g)$

The asymptotic series suffer from Borel singularities and require a regularization

Resurgence

Borel transform

$$\mathcal{B}(s) = \sum_{n=0}^{\infty} \alpha_n \frac{s^n}{\Gamma(n+1)}$$

$$f_{\text{oct}}(g) = 4\pi g \int_0^\infty ds\, e^{-4\pi g s} \mathcal{B}(s)$$

The function $\mathcal{B}(s)$ has poles which condense on the real axis for $s > 2$

$$\mathcal{B}(s) \sim \frac{1}{s-2} - \log(2-s) \sum_{k\geq 0} b_{k+1}^{(1)} \frac{(s-2)^k}{k!}$$

Regularize the integral by deforming the integration contour slightly above or below the poles

$$f_{\pm}(g) = 4\pi g \int_0^{\infty e^{\pm i\epsilon}} ds\, e^{-4\pi g s} \mathcal{B}(s)$$

Ambiguity introduced by the Borel singularities

$$f_+(g) - f_-(g) = 4\pi g \int_0^\infty ds\, e^{-4\pi g s}\, \text{disc}\, \mathcal{B}(s) \sim 4\pi g\, e^{-8\pi g} + O(e^{-16\pi g})$$

What is a meaning of this ambiguity?

Resurgence relations for the octagon

Ambiguities generated by Borel singularities at $s > 0$ should cancel in the sum $f_{\text{oct}}(g) + \Delta f_{\text{oct}}(g)$

This leads to nontrivial relations between large-order behaviour of the perturbative coefficients

$$\alpha_{n\gg 1} \sim \sum_{k\geq 0} b_k^{(1)} \frac{(n-k)!}{2^{n+1-k}} + b_k^{(2)} \frac{(n+1-k)!}{4^{n+2-k}} + \cdots$$

and the nonperturbative coefficient functions

$$\Delta f_{\text{oct}}(g) = \frac{i\pi g'}{4} e^{-8\pi g} \left[1 + \frac{c_1}{(4\pi g')} + \frac{c_2}{(4\pi g')^2} + \ldots\right] + O(e^{-16\pi g})$$

Resurgence relations

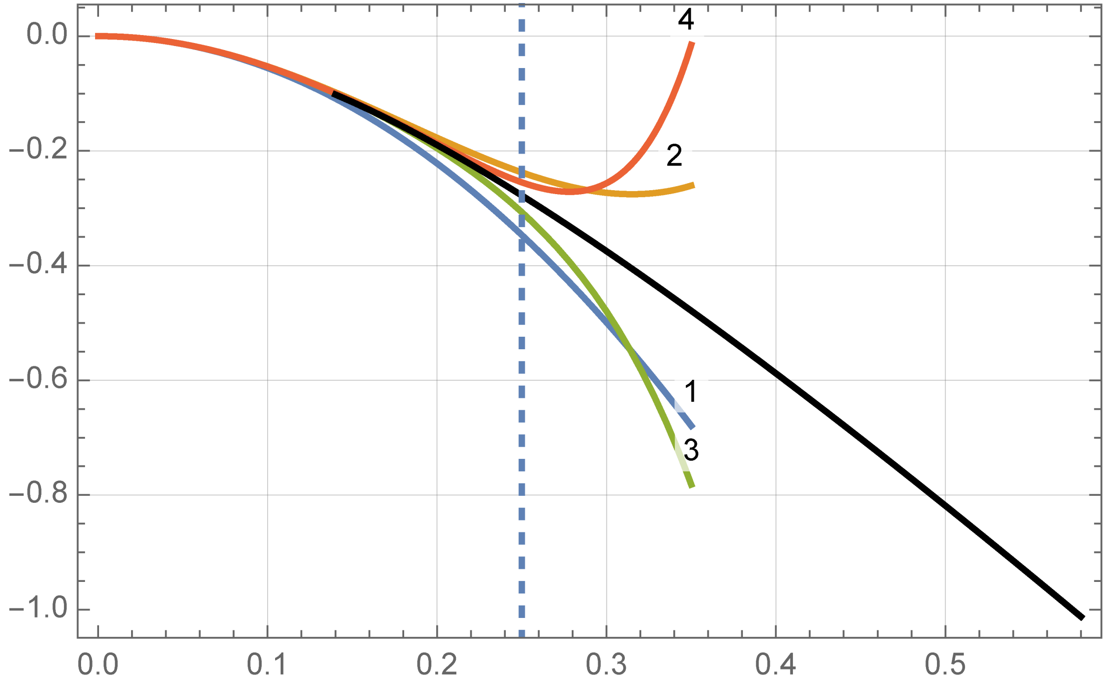
$$c_1 = b_1^{(1)}/b_0^{(1)}\,, \qquad\qquad c_2 = b_1^{(2)}/b_0^{(1)}\,, \qquad\qquad \ldots$$

High-precision numerical calculation of $\alpha_{n\leq 400}$ gives $b_1^{(1)}/b_0^{(1)} = -7/4$ and $b_2^{(1)}/b_0^{(1)} = -63/32$

Perfect agreement with the analytical result

$$\Delta f_{\text{oct}}(g) = \frac{i\pi g'}{4} e^{-8\pi g} \left[1 - \frac{\frac{7}{4}}{(4\pi g')} - \frac{\frac{63}{32}}{(4\pi g')^2}\right] + \ldots$$

Octagon at arbitrary coupling constant



Dependence of the octagon on ’t Hooft coupling constant $g = \sqrt{\lambda}/(4\pi)$

Colored curves represent the first few terms in the weak coupling expansion

The black curve describes the strong coupling expansion defined by the latteral Borel summation

The vertical dashed line indicates the convergence radius of the weak coupling expansion

Open questions

Various quantities (free energy, correlation functions, Wilson loop, tilted cusp) in *different* 4d super Yang-Mills theories are expressed in terms of the *same* (temperature dependent) Tracy-Widom distribution

This relation is powerful enough to predict the dependence on 't Hooft coupling

- ✔ Who ordered this universality?
- ✔ What is the reason why the Bessel kernel appears in all cases?
- ✔ How to reproduce the strong coupling expansion from holography?
- ✔ What is the meaning of the nonperturbative scales $\Lambda^2 = g^a e^{-8\pi g x_1}$ in AdS/CFT?

Many thanks from all of us to Yunfeng and Xinan for organizing such a successful workshop!