Modular structures in $\mathcal{N} = 4$ susy Yang-Mills theory

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Joint work w/
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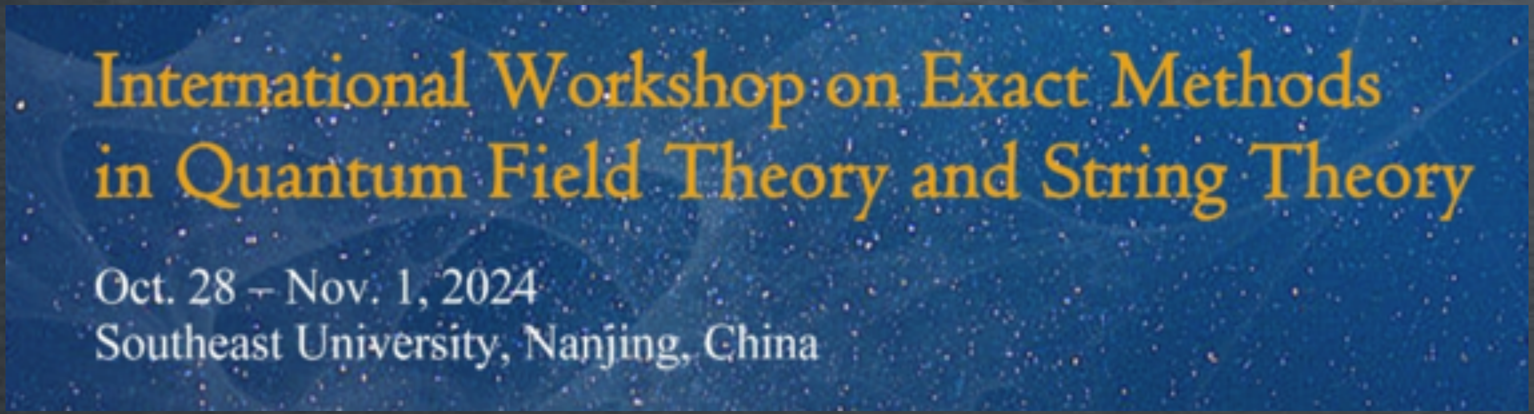
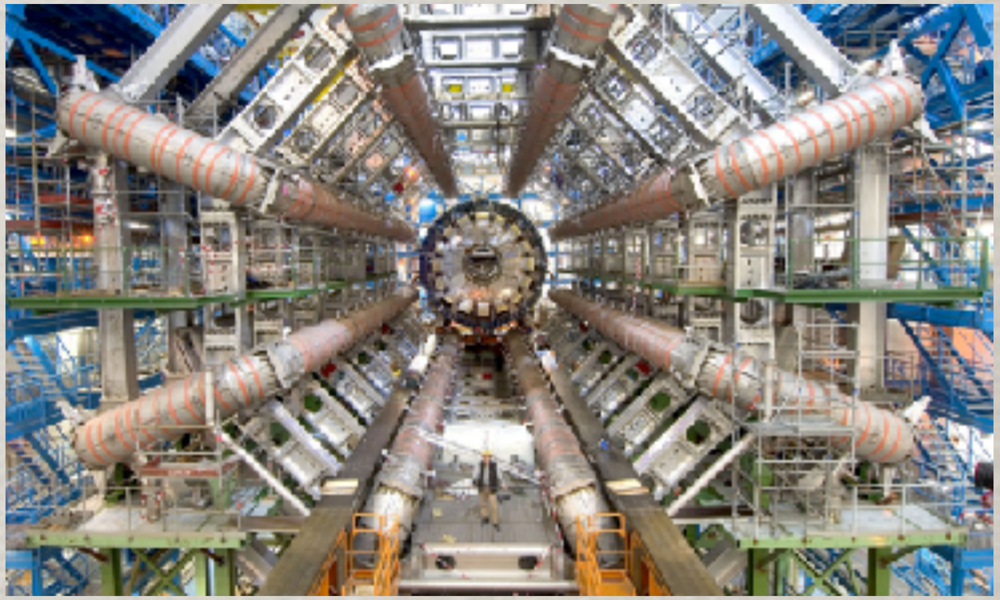


October 29th, 2024

$\mathcal{N} = 4$ SUSY Yang-Mills Theory

Beautiful "Lab" to extract exact results, e.g. correlation fcts
Uniquely specified by:



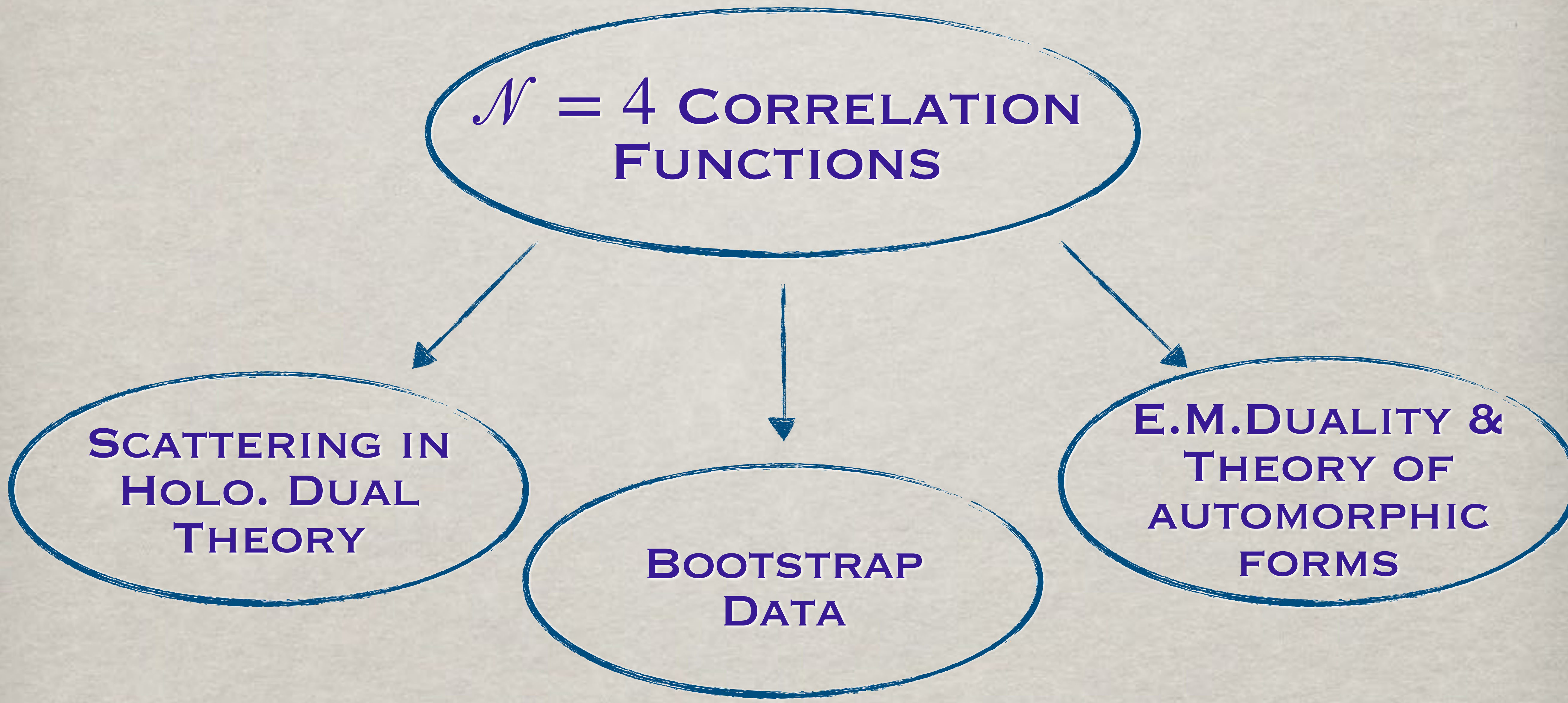
$\mathcal{N} = 4$ SYM Data:

- 4d Gauge theory: group G

  In this talk: G=SU(N)

- single marginal deformation, coupling constant τ

$$\tau = \tau_1 + i\tau_2 = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{\text{YM}}^2} \in \mathbb{H}$$


$\mathcal{N} = 4$ Correlation Functions
Scattering in Holo. Dual Theory
Bootstrap Data
E.M.Duality & Theory of automorphic forms

What type of correlators

$\mathcal{N} = 4$ correlators

Amongst the many fields of $\mathcal{N} = 4$ we have:

Φ_I I=1,...,6 Adjoint scalar

$$\mathcal{O}_2(x, Y) = \mathrm{Tr}(\Phi_I \Phi_J) Y^I Y^J \qquad \Delta = 2$$

Null polarisation vectors for

1/2 BPS operators, super-conformal primaries in the stress-energy tensor super-multiplet ([0,2,0])

$\mathcal{N} = 4$ CORRELATORS

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1/2 BPS operators, super-conformal primaries in the stress-energy tensor super-multiplet

We consider the (NON-PROTECTED!) 4pt function

$$\langle \mathcal{O}_2(x_1, Y_1) ... \mathcal{O}_2(x_4, Y_4) \rangle = \frac{1}{|x_{12}|^4 |x_{34}|^4} \mathcal{I}(Y) \mathcal{T}_{G_N}(u, v)$$

$\mathcal{N} = 4$ correlators

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Fixed by super-conformal symmetry

$\mathcal{N} = 4$ CORRELATORS

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Non-trivial function of u,v, coupling, gauge group G_N

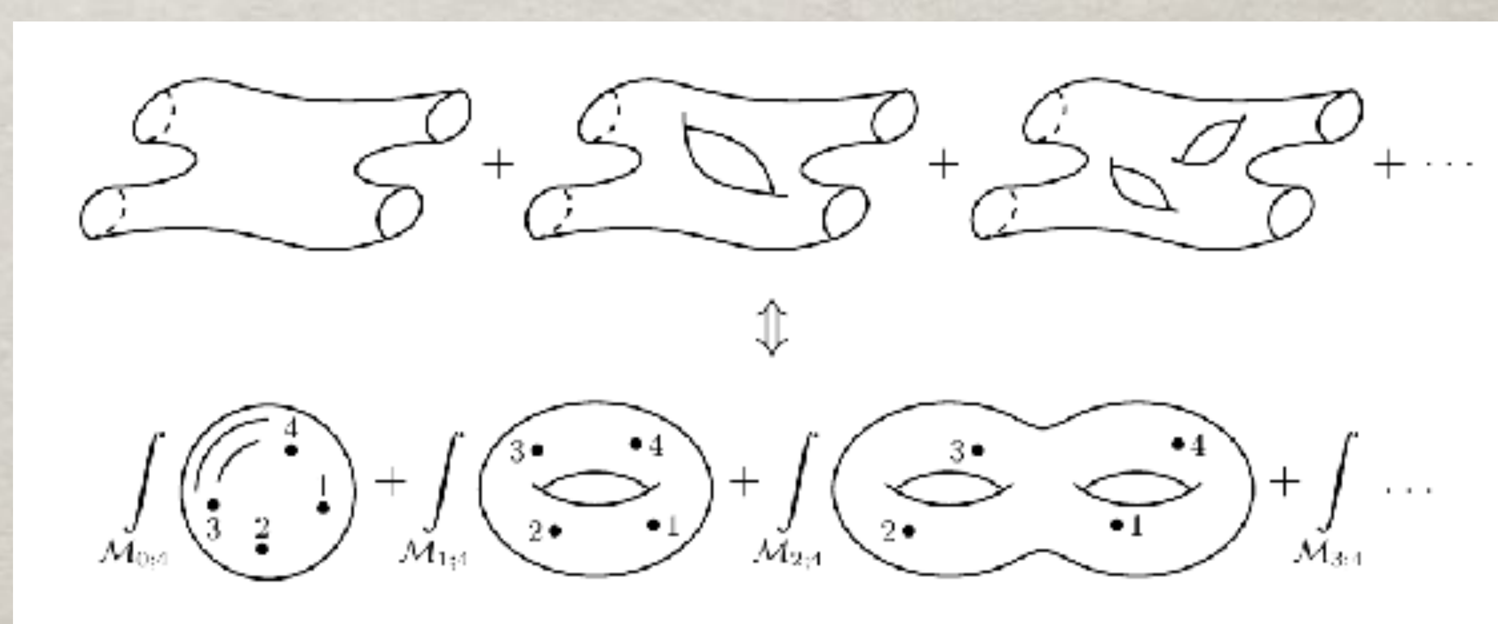
Why do we care?

stress-energy tensor $\simeq$ gravitons

$$\langle \mathcal{O}_2(x_1, Y_1) \ldots \mathcal{O}_2(x_4, Y_4) \rangle = \frac{1}{|x_{12}|^4 |x_{34}|^4} \mathcal{I}(Y) \mathcal{T}_{G_N}(u, v)$$

Fun part = hard part

$$\langle T_{\mu_1 \nu_1}(x_1) \ldots T_{\mu_4 \nu_4}(x_4) \rangle$$



4-graviton amplitude in IIB when $G_N = SU(N)$ @large-N

How to get a handle on

$\mathcal{N} = 4$ Integrated correlators

Pestun' Susy localisation for $\mathcal{N} = 2^*$ partition function on S^4
$Z_G(m; \tau)$; massive deformation, m, of $\mathcal{N} = 4$

Path-Integral $\Longrightarrow$ Matrix Model

$$Z_G(m, \tau) := \int V_G(a) e^{-2\pi\tau_2 \langle a,a \rangle} \hat{Z}_G^{\text{pert}}(m; a) |\hat{Z}_G^{\text{inst}}(m, \tau; a)|^2 d^r a$$

1-Loop   Nekrasov

$$\tau = \tau_1 + i\tau_2 = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{YM}^2}$$

G Gauge group

Perturbative
1-Loop determinant

Instanton
partition function

$\mathcal{N} = 4$ Integrated correlators

Pestun' Susy localisation for $\mathcal{N} = 2^*$ partition function on S^4
$Z_G(m;\tau)$; massive deformation, m, of $\mathcal{N} = 4$

$$\mathcal{C}_G(\tau) = \frac{1}{4}\Delta_\tau \partial_m^2 \log Z_G(m;\tau)|_{m=0}$$

[Binder, Chester, Pufu, Wang]

$$= \int d\mu(\{x_i\}) \langle \mathcal{O}_2(x_1) \cdots \mathcal{O}_2(x_4) \rangle$$

$$\mathcal{H}_G(\tau) = \partial_m^4 \log Z_G(m;\tau)|_{m=0}$$

$$= \int d\tilde{\mu}(\{x_i\}) \langle \mathcal{O}_2(x_1) \cdots \mathcal{O}_2(x_4) \rangle$$

[Chester, Pufu]

Very specific measures fixed by susy!

Line-Defect Integrated correlators

Integrated Correlator with line-defects:

[Pufu, Rodriguez, Wang]
[Billo', Galvagno, Frau, Lerda]

$$I_{\mathbb{W}}(N;\tau) = \partial_m^2 \log W_{SU(N)}(m,\tau)|_{m=0}$$
$$= \int \mathrm{d}\nu(\{x_i\}) \langle \mathbb{W} \mathcal{O}_2(x_1) \mathcal{O}_2(x_2) \rangle$$

1/2-BPS fundamental Wilson loop along great circle of S^4 equivalently straight-line Wilson loop on $\mathbb{R}^4$

$\mathcal{N} = 4$ Integrated correlators

For the rest of the talk G=SU(N):
We will discuss the integrated correlators

$\mathcal{C}(N;\tau)$ → First integrated Correlator

$\mathcal{H}(N;\tau)$ → Second integrated Correlator

$\mathcal{I}_{\mathbb{L}}(N;\tau)$ → Defect integrated Correlator

Key Message(s):

Electro-magnetic (Olive-Montonen/GNO) duality strongly constraints $\mathcal{N} = 4$ SYM observables!

$\mathcal{N} = 4$ Integrated correlators

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Key Message(s):

- non-holomorphic functions of coupling τ: **Lattice sums!**
- "nice" automorphic properties under

$$\tau \to \gamma \cdot \tau = \frac{a\tau + b}{c\tau + d} \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

First Integrated Correlator

First Integrated correlator

From matrix model localised partition function on S^4

$$\mathcal{C}_G(\tau) = \frac{1}{4}\Delta_\tau \partial_m^2 \log Z_G(m;\tau)|_{m=0}$$

[Binder,Chester,Pufu,Wang]

$$= \int \mathrm{d}\mu(\{x_i\})\langle \mathcal{O}_2(x_1)...\mathcal{O}_2(x_4)\rangle$$

We found an exact formula valid
for all simple gauge groups G
and all values of the coupling τ and GNO co/in-variant

[DD,Green,Wen]
[DD,Vallarino]

Using key-results of [Chester, Green, Pufu, Wang, Wen]
and [Alday,Chester,Hansen]

First Integrated correlator SU(N)

Lattice sum representation for G=SU(N),

$$\mathcal{C}(N;\tau) = \sum_{(m,n)\in\mathbb{Z}^2} \int_0^\infty e^{-t\frac{\pi|m+n\tau|^2}{\tau_2}} B(N;t)\mathrm{d}t$$

- Very unexpected (from matrix model) lattice sum!
- Modular invariant (or Fricke groups) function of τ
- Proof using [Harer, Zagier]
- $B(N;t)$ rational fcts expressed in terms of Jacobi Poly

e.g. $B(2;t) = \dfrac{3(3t - 10t^2 + 3t^3)}{2(1+t)^5}$

Resurgent large-N expansion of the First integrated Correlator

SU(N) @ Large N

[DD,Green,Wen,Xie]-[DD,Treilis]

$$\mathcal{C}_N(\tau) = \sum_{(m,n)\in\mathbb{Z}^2} \int_0^\infty e^{-t\pi\frac{|m+n\tau|^2}{\tau_2}} B_{SU(N)}(t)\,\mathrm{d}t$$

@Large-N <u>Fixed-τ</u>: split into P and NP contributions (<u>in N</u>)

$$\mathcal{C}_N(\tau) = \mathcal{C}_N^{P}(\tau) + \mathcal{C}_N^{NP}(\tau)$$

$$\mathcal{C}_N^{P}(\tau) = \frac{N^2}{4} + \sum_{r=0}^{\infty} N^{\frac{1}{2}-r} f_r(\tau)$$

$$\mathcal{C}_N^{NP}(\tau) = O(N^2 e^{-\sqrt{N}})$$

Large-N Perturbative

[DD,Green,Wen,Xie]-[DD,Treilis]

$$\mathcal{C}_N(\tau) = \mathcal{C}_N^P(\tau) + \mathcal{C}_N^{NP}(\tau)$$

The large-N expansion changes dramatically only <u>half-integer</u> index Eisensteins appear:

$$\mathcal{C}_N(\tau) \sim \mathcal{C}_N^P(\tau) = \frac{N^2}{4} - \frac{3N^{\frac{1}{2}}}{2^4}E(\tfrac{3}{2};\tau) + \frac{45}{2^8 N^{\frac{1}{2}}}E(\tfrac{5}{2};\tau) + \frac{3}{N^{\frac{3}{2}}}\Big[\frac{1575}{2^{15}}E(\tfrac{7}{2};\tau) - \frac{13}{2^{13}}E(\tfrac{3}{2};\tau)\Big] + O(N^{-\frac{5}{2}})$$

Non-holo (Real analytic) Eisensteins Series

$$E^*(s;\tau) = \frac{\Gamma(s)}{2} \sum_{(m,n)\neq(0,0)} \frac{(\tau_2/\pi)^s}{|m+n\tau|^{2s}} = E^*(1-s;\tau)$$

$$= \xi(2s)\tau_2^s + \xi(2s-1)\tau_2^{1-s}$$

$$+ \sum_{k\neq 0} e^{2\pi i k\tau_1} 2\sqrt{\tau_2}|k|^{s-\frac{1}{2}} \sigma_{1-2s}(k) K_{s-\frac{1}{2}}(2\pi|k|\tau_2)$$

<u>Modular invariant functions:</u>

$$E^*(s;\gamma\cdot\tau) = E^*(s;\tau)\,, \qquad \forall\gamma\in SL(2,\mathbb{Z})$$

<u>Laplace eigenfunctions:</u>

$$(\Delta_\tau - s(s-1))E^*(s;\tau) = 0$$

Holographic Picture

stress-energy tensor ≃ gravitons

$$\langle \mathcal{O}_2(x_1, Y_1) \ldots \mathcal{O}_2(x_4, Y_4) \rangle = \frac{1}{|x_{12}|^4 |x_{34}|^4} \mathcal{I}(Y) \mathcal{T}_{G_N}(u, v)$$

↓

$$\langle T_{\mu_1\nu_1}(x_1) \ldots T_{\mu_4\nu_4}(x_4) \rangle$$

↓

4-graviton amplitudes in IIB

$$\tau = \chi + i/g_s$$

Axio-dilaton

$$\frac{(\alpha')^2}{L^4} = \frac{1}{N g_{_{YM}}^2}$$

Large-N Perturbative

-Fixed g_{YM} large-N (modularity is preserved):

[Chester, Green, Pufu, Wang, Wen]

$$\mathcal{C}_{SU(N)}(\tau,\bar\tau) \sim \frac{N^2}{4} - \frac{3N^{\frac{1}{2}}}{2^4} E(\tfrac{3}{2};\tau,\bar\tau) + \frac{45}{2^8 N^{\frac{1}{2}}} E(\tfrac{5}{2};\tau,\bar\tau)$$
$$+ \frac{3}{N^{\frac{3}{2}}}\left[\frac{1575}{2^{15}} E(\tfrac{7}{2};\tau,\bar\tau) - \frac{13}{2^{13}} E(\tfrac{3}{2};\tau,\bar\tau)\right] + O(N^{-\frac{5}{2}})$$

$$\tau = \chi + i/g_s$$

$$\frac{(\alpha')^2}{L^4} = \frac{1}{N g_{_{YM}}^2}$$

4-graviton effective action in type IIB low-energy expansion

[Green, Gutperle - Green, Vanhove- Green, Miller, Vanhove]

$$\mathcal{L}_{eff} = (\alpha')^{-4} g_s^{-2} R + f_1(\tau,\bar\tau)(\alpha')^{-1} g_s^{-1/2} R^4 + f_2(\tau,\bar\tau)\alpha' g_s^{1/2} d^4 R^4 + f_3(\tau,\bar\tau)(\alpha')^2 g_s d^6 R^4 + \ldots$$

N^2 $\qquad N^{1/2}$ $\qquad N^{-1/2}$ $\qquad N^{-1}$

Modular Resurgence @ Large-N

@Large-N Perturbative expansion divergese factorially!

[DD, Green, Wen, Xie]- [DD, Treilis]

$$\mathcal{C}_N(\tau) = \mathcal{C}_N^P(\tau) + \mathcal{C}_N^{NP}(\tau)$$

Resurgence analysis:
infinite tower of modular invariant NP corrections
from Perturbative data!!

$$\mathcal{C}_N^{NP}(\tau) = -2N^2 D_N(0;\tau) + N^{\frac{3}{2}}\left[\tfrac{1}{3} D_N(-\tfrac{3}{2};\tau) - \tfrac{9}{4} D_N(\tfrac{1}{2};\tau)\right] + O(N)$$

$$D_N(s;\tau) = \sum_{(m,n)\neq(0,0)} \exp\left(-4\sqrt{\frac{N|m+n\tau|^2}{\tau_2}}\right)\frac{\tau_2^s}{|m+n\tau|^{2s}}$$

Large-N Non-Perturbative

Novel NP modular invariant functions! [DD,Green,Wen,Xie]

$$D_N(s;\tau) = \sum_{(m,n)\neq(0,0)} \exp\Big(-4\sqrt{\frac{N|m+n\tau|^2}{\tau_2}}\Big)\frac{\tau_2^s}{|m+n\tau|^{2s}}$$

- Arise also in large-charge expansion of $\mathcal{O}_p$ integrated corr.
  [Paul,Perlmutter,Raj]-[Brown,Wen Xie]

- Torodial Casimir energy in 3-dimensional CFTs
  [Luo,Wang]

<u>In 't Hooft limit:</u> $\lambda = 4\pi N/\tau_2$ fixed / $\tilde{\lambda} = N^2/\lambda$ fixed

F-string world-sheet instantons $e^{-2\ell\sqrt{\lambda}}$ and

"dyonic" instantons $e^{-2\ell\sqrt{\tilde{\lambda}}}$

Reproducing resurgence results at large-λ, large-$\tilde{\lambda}$

[DD,Green,Wen] - [Collier,Perlmutter] - [Hatsuda,Okuyama]

Holographic Interpretation:

Holo. Dictionary: consider $AdS_5 \times S^5$ with scale L

$$g_{YM}^2 = \frac{4\pi}{\tau_2} = 4\pi g_s \qquad \text{and} \qquad \sqrt{g_{YM}^2 N} = \frac{L^2}{\alpha'}$$

$$T_F = \frac{1}{2\pi\alpha'} \qquad \Rightarrow \qquad T_{p,q} = T_F |p + q\tau|$$

$$\mathcal{C}^{N.P.}_{SU(N)}(\tau, \bar{\tau}) \to \sum_{\ell=1}^{\infty} \sum_{\gcd(p,q)=1} \exp\Big(-4\pi L^2 \ell\, T_{p,q} \Big)$$

NP terms are given by sum over ℓ coincident (p,q)-strings euclidean world-sheet wrapping a great S^2 inside S^5

[Some key differences for SO and USp]

Defect integrated correlator

Second integrated correlator

First integrated correlator

large-N expansion of the <u>Second</u> integrated Correlator

Second integrated correlator

[Alday, Chester, DD, Green, Wen]

$$\mathcal{H}(N;\tau) = \partial_m^4 \log Z_{SU(N)}(m;\tau)|_{m=0}$$

[Chester,Pufu]

$$= \int \mathrm{d}\tilde{\mu}(\{x_i\})\langle \mathcal{O}_2(x_1)...\mathcal{O}_2(x_4)\rangle$$

<u>Large-N Perturbative expansion:</u>

$$\mathcal{H}(N;\tau) \sim 6N^2 + \mathcal{H}^h(N;\tau) + \mathcal{H}^i(N;\tau)$$

Second integrated correlator

[Alday, Chester, DD, Green, Wen]

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$$\mathcal{H}(N;\tau) \sim 6N^2 + \mathcal{H}^h(N;\tau) + \mathcal{H}^i(N;\tau)$$

Half-integer powers in 1/N $\longrightarrow$ Same structure as first Integrated Correlator

Can find NP completion with Modular/Resurgence

[DD,Treilis]

Second integrated correlator

[Alday, Chester, DD, Green, Wen]

$$\mathcal{H}(N;\tau) \sim 6N^2 + \mathcal{H}^h(N;\tau) + \mathcal{H}^i(N;\tau)$$

Integer powers in 1/N $\longrightarrow$ Generalised Eisenstein Series

Modular invariant solutions to inhomogeneous Laplace eq.

$$\big(\Delta_\tau - s(s-1)\big)\mathcal{E}(s;s_1,s_2;\tau) = E(s_1;\tau)E(s_2;\tau)$$

e.g. higher derivative correction d^6R^4 in IIB: $\mathcal{E}(4;\frac{3}{2},\frac{3}{2};\tau)$

[Green, Miller, Vanhove]

$$\mathcal{L}_{eff} = (\alpha')^{-4}g_s^{-2}R + f_1(\tau,\bar\tau)(\alpha')^{-1}g_s^{-1/2}R^4 + f_2(\tau,\bar\tau)\alpha' g_s^{1/2}d^4R^4 + f_3(\tau,\bar\tau)(\alpha')^2 g_s d^6R^4 + \ldots$$

Second integrated correlator

[Alday, Chester, DD, Green, Wen]

Order by order in 1/N : $\mathcal{H}_N^i(\tau)$ Lattice sum!!

$p_i \in \mathbb{Z} + \tau\mathbb{Z}$

$$\mathcal{E}^w_{i,j}(\tau) = \sum_{\substack{p_1,p_2,p_3\neq 0\\ p_1+p_2+p_3=0}} \int_0^\infty d^3t\, B^w_{i,j}(t_1,t_2,t_3)\, \exp\Big(-\frac{\pi}{\tau_2}\sum_{i=1}^3 t_i|p_i|^2\Big)$$

Modular Local Harmonic Maass-forms

[Zagier], see also [Bringmann,Kane] and to appear [DD,Green,Wen]

Special rational linear combinations of generalised Eisenstein series for which L-values of holomorphic cusp forms drops out

[DD,Kleinschmidt,Schlotterer]-[Fedosova,Klinger-Logan,Radchenko]

On the usefulness of integrated correlators

integrated correlator Constraints

Both first and second integrated correlators come from <u>SAME</u> four point function!

$$\mathcal{C}_N(\tau) = \int \mathrm{d}\mu(\{x_i\}) \langle \mathcal{O}_2(x_1)...\mathcal{O}_2(x_4)\rangle$$

$$\mathcal{H}_N(\tau) = \int \mathrm{d}\tilde{\mu}(\{x_i\}) \langle \mathcal{O}_2(x_1)...\mathcal{O}_2(x_4)\rangle$$

Very specific measures fixed by susy!

$$\mathrm{d}\mu(\{x_i\}) = \frac{r^3 \sin^2(\theta)}{U^2}$$

$$U = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} = 1 + r^2 - 2r\cos(\theta)$$

$$V = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} = r^2$$

Integrated correlator Constraints

Both first and second integrated correlators come from <u>SAME</u> four point function!

$$\mathcal{C}_N(\tau) = \int \mathrm{d}\mu(\{x_i\}) \langle \mathcal{O}_2(x_1) \ldots \mathcal{O}_2(x_4) \rangle$$

$$\mathcal{H}_N(\tau) = \int \mathrm{d}\tilde{\mu}(\{x_i\}) \langle \mathcal{O}_2(x_1) \ldots \mathcal{O}_2(x_4) \rangle$$

Exact & NP
from susy loc.

$$= \sum_{\Delta,\ell} |C_{\Delta,\ell}|^2 \mathcal{F}_{\Delta,\ell}(u,v)$$

Superconformal block
decomposition

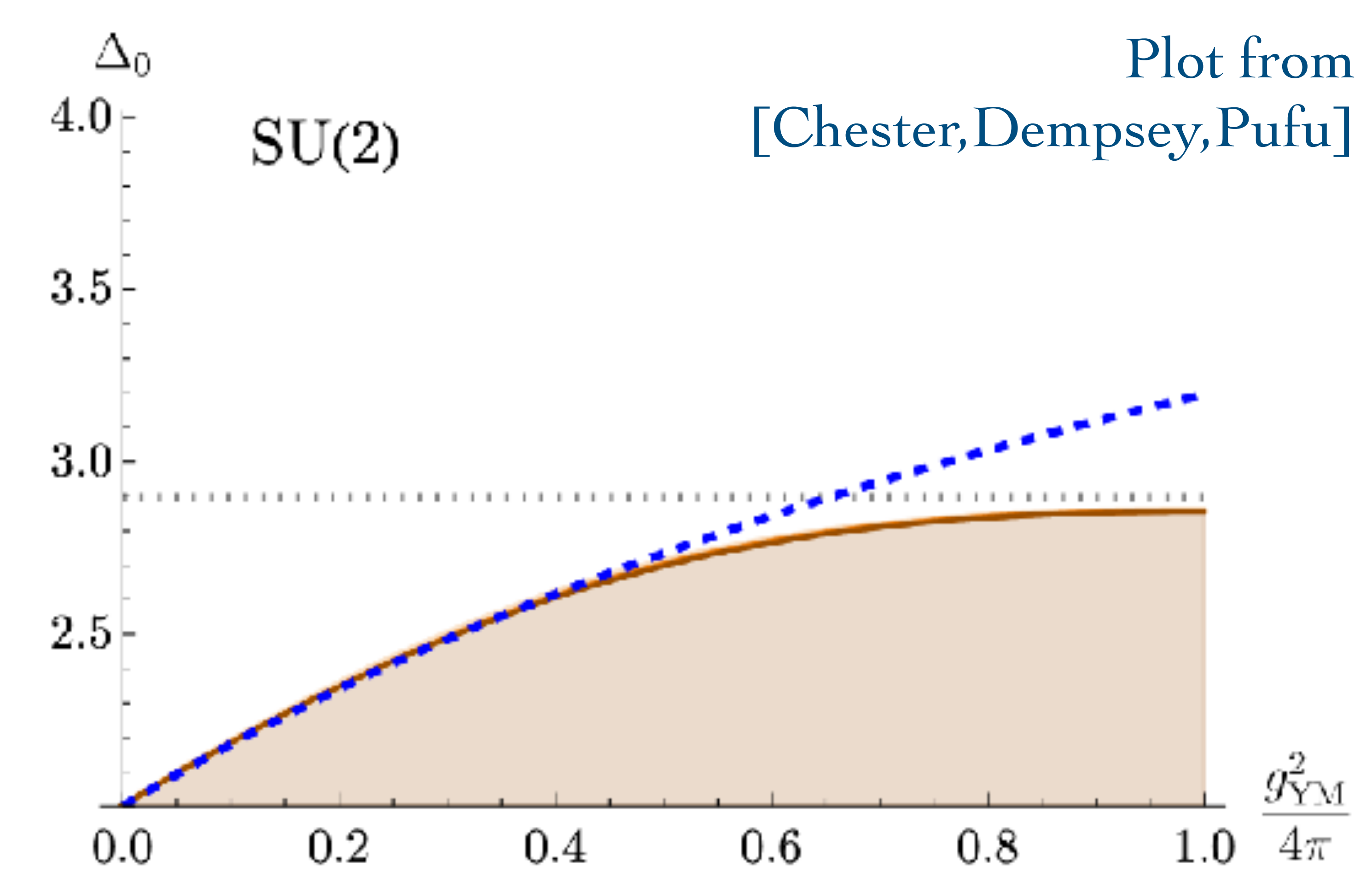
INTEGRATED CORRELATOR CONSTRAINTS

Both first and second integrated correlators come from <u>SAME</u> four point function!

$$\mathcal{C}_N(\tau) = \int \mathrm{d}\mu(\{x_i\})\langle \mathcal{O}_2(x_1)...\mathcal{O}_2(x_4)\rangle$$

$$\mathcal{H}_N(\tau) = \int \mathrm{d}\tilde{\mu}(\{x_i\})\langle \mathcal{O}_2(x_1)...\mathcal{O}_2(x_4)\rangle$$

Superconformal Bootstrap
aided by integrated correlators



Plot from [Chester,Dempsey,Pufu]

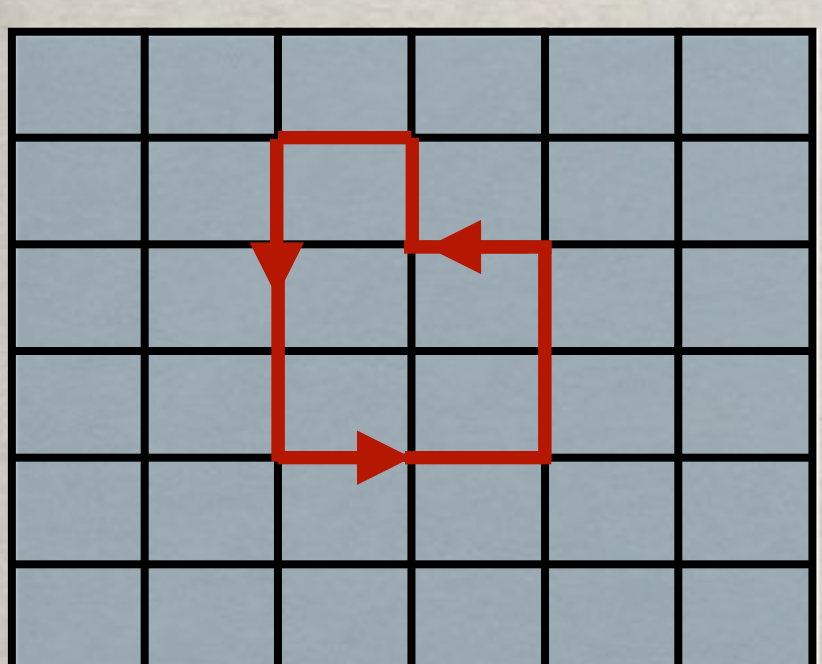
Line-defect integrated Correlators

Importance of Line-Defects:

Important examples of non-local operators:

- QCD: confinement/deconfinement phase transition;
- important for higher form symmetries;
- class of $\mathcal{N} = 4$ SYM line defect integrated correlators

  teaches us about graviton scattering from branes.



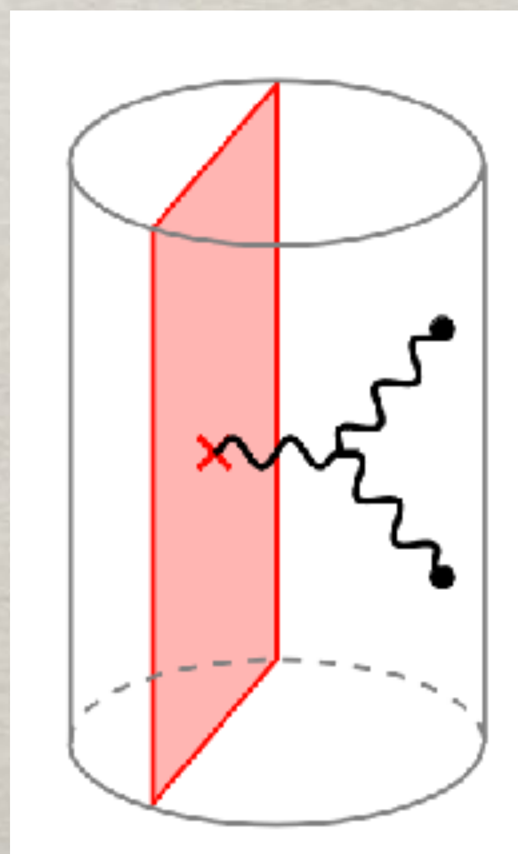


Figure from [Pufu,Rodriguez,Wang]

Line-Defects & EM Duality:

We consider a line defect $\mathbb{L}$ in $\mathcal{N} = 4$ SU(N) SYM parametrised by an electric and a magnetic charge: $p, q \in \mathbb{Z}$ with $(p, q) = 1$

- $(p, q) = (1,0) \longrightarrow$ Wilson loop $\mathbb{W}$
- $(p, q) = (0,1) \longrightarrow$ 't Hooft loop $\mathbb{T}$

Under electromagnetic duality (Olive-Montonen/GNO):

$$\tau \to \tau' = \gamma \cdot \tau = \frac{a\tau + b}{c\tau + d} \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

$$(p, q) \to (p', q') = (p, q)\begin{pmatrix} a & -c \\ -b & d \end{pmatrix}$$

$$\mathbb{L}_{p,q} \longrightarrow \mathbb{L}_{p',q'}$$

Line-Defects & EM Duality:

Line-defect integrated correlator:

[Pufu, Rodriguez, Wang]
[Billo', Galvagno, Frau, Lerda]

$$I_{\mathbb{W}}(N;\tau) = \partial_m^2 \log W_{SU(N)}(m,\tau)|_{m=0}$$
$$= \int \mathrm{d}\nu(\{x_i\})\langle \mathbb{W}\mathcal{O}_2(x_1)\mathcal{O}_2(x_2)\rangle$$

↓

1/2-BPS fundamental Wilson loop along great circle

$\mathcal{N} = 4$ SYM EM duality constraint:

$$I_{\mathbb{W}}(N;\gamma\cdot\tau) = I_{\mathbb{L}_{p,q}}(N;\tau)$$

$$\gamma\cdot\tau = \frac{a\tau+b}{q\tau+p}$$

$$\gamma = \begin{pmatrix} a & b \\ q & p \end{pmatrix} \in SL(2,\mathbb{Z})$$

e.g. $I_{\mathbb{W}}(N;-\frac{1}{\tau}) = I_{\mathbb{T}}(N;\tau)$

Line-Defects & EM Duality:

Useful labelling of a line-defect $\mathbb{L}$ as:

$[\rho] \in B(\mathbb{Z})\backslash SL(2,\mathbb{Z}) \simeq \{(p,q) \in \mathbb{Z}^2 \,|\, (p,q) = 1\,, q \geq 0\}$

$[\rho] = \begin{pmatrix} * & * \\ q & p \end{pmatrix}$

- $(p,q) = (1,0) \longrightarrow$ Wilson loop $\mathbb{W}$ $\rightarrow \quad [\rho] = [\mathbb{1}]$
- $(p,q) = (0,1) \longrightarrow$ 't Hooft loop $\mathbb{T}$ $\rightarrow \quad [\rho] = [S] = [\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}]$

<u>$\mathcal{N} = 4$ SYM EM duality expectation:</u>

$$I_{\mathbb{L}}([\rho];\tau) = I_{\mathbb{L}'}([\rho\gamma^{-1}];\gamma\cdot\tau)\,, \qquad \forall\gamma \in SL(2,\mathbb{Z})$$

e.g. $$I_{\mathbb{W}}(-\frac{1}{\tau}) = I_{\mathbb{L}}([\mathbb{1}];S\cdot\tau) = I_{\mathbb{L}'}([S];\tau) = I_{\mathbb{T}}(\tau)$$

LINE-DEFECTS & EM DUALITY:

[DD,Duan, Pavarini, Xie, Wen]

$\mathcal{N} = 4$ SYM integrated correlator of a line defect $\mathbb{L}$ with charges (p, q)

- lattice-sum representation
- novel automorphic functions:

$$F_{\mathbb{L}_{p,q}}(s_1, s_2, s_3; \tau) = \frac{\tau_2^{s_1}}{|q\tau + p|^{2s_1}} \sum_{(n,m)\neq\mathbb{Z}(q,p)} \frac{\tau_2^{s_2}}{|n\tau + m|^{2s_2}} (np - mq)^{s_3}$$

$$s_1, s_2 \in \mathbb{Z} \text{ or } \mathbb{Z} + \frac{1}{2}$$

$$s_3 \in 2\mathbb{Z}$$

→ Finite-N/Large-N transition

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<u>Importantly:</u>

<u>E/M Duality:</u>

$$F_{\mathbb{L}_{p,q}}(s_1, s_2, s_3; \tau) = F_{\mathbb{L}_{p',q'}}(s_1, s_2, s_3; \gamma \cdot \tau)$$

$$\tau \to \tau' = \gamma \cdot \tau$$

$$(p, q) \to (p', q') = (p, q)\begin{pmatrix} a & -c \\ -b & d \end{pmatrix}$$

Line-Defects & EM Duality:

@ large N and fixed τ :

$$\mathcal{I}_{\mathbb{W},N}(\tau) = \sum_{\ell=-1}^{\infty} N^{-\ell/2} \mathcal{I}_{\mathbb{W}}^{(\ell)}(\tau)$$

The coefficient of each 1/N order is given by a finite, rational linear combinations of $F_{\mathbb{W}}(s_1, s_2, s_3; \tau)$

@ finite N and fixed τ :

$$\mathcal{I}_{\mathbb{L},N}(p,q;\tau) = \frac{N}{L^1_{N-1}(-\frac{\pi|q\tau+p|^2}{\tau_2})} \sum_{(n,m)\in\mathbb{Z}^2} \int_0^\infty e^{-t_1 \frac{\tau_2}{\pi|q\tau+p|^2}} e^{-t_2\pi \frac{|n\tau+m|^2}{\tau_2}} e^{-t_3\pi \frac{\tau_2}{|q\tau+p|^2}(np-mq)^2} \mathcal{B}_N(t_1,t_2,t_3)\, \mathrm{d}^3 t$$

$$\mathcal{I}_{\mathbb{L}_{p,q},N}(\tau) \text{ ``$=$'' } \sum_{s_1,s_2,s_3=1}^{\infty} d^{(N)}_{s_1,s_2,s_3} F_{\mathbb{L}_{p,q}}(s_1,s_2,s_3;\tau)$$

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E.g. We can predict the exact 't Hooft line-defect integrated correlator in SU(N) $\mathcal{N} = 4$ SYM

[Dorigoni]


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Conclusions:

- **<u>The power of modularity</u>:** Astonishingly simple & beautiful non-perturbative results for non-protected correlators in $\mathcal{N} = 4$ SYM!

- Integrated correlators and super conformal bootstrap
  [Chester,Dempsey,Pufu] see also [Behan,Chester,Ferrero]

- <u>Other integrated correlators</u>:
  - Higher point functions MUV [DD,Green,Wen]
  - Higher-charge operators [Brown,Wen, Xie]-[Paul,Perlmutter,Raj]
  - Giant-graviton operators [Brown,Galvagno,Wen]
  - $\mathcal{N} = 2$ SYM [Billò,Frau,Lerda,Pini,Vallarino]-[Pini,Vallarino]

Open Questions:

- Systematic of finite-N/large-N for second correlator & defect correlator?
- Defect conformal bootstrap from integrated correlators?
- String theory/QFT origin of these lattice-sum representations?
- What is the origin of these differential structures? 2d/4d correspondence?
- **Math.NT**: Why does string theory only like MZVs?
- How far can we push integrated correlators to learn about un-integrated stuff?

Thank you for your attention!