量子场论与弦论中的精确方法国际研讨会

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Witten Indices of D0-Brane Quantum Mechanics

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Outline

- Bank-Fischler-Shenker-Susskind (BFSS) and Berenstein-Maldacena-Nastase (BMN) matrix quantum mechanics
- Holography 1: BFSS conjecture
- Holography 2: gauge/gravity duality
- Counting black hole microstates via BMN

There is a very nice review talk on the BFSS conjecture by Juan Maldacena at Strings2024. slides, video

Some parts of this talk are taken from this review.

BFSS Matrix Quantum Mechanics

$$S = \frac{1}{g_{\rm YM}^2}\int dt\,{\rm Tr}\left[\frac{1}{2}\sum_{I=1}^{9}(D_tX^I)^2 + \frac{1}{4}\sum_{I,J=1}^{9}[X^I, X^J]^2 + \frac{1}{2}\psi^\alpha D_t\psi^\alpha + \frac{1}{2}\psi^\alpha\Gamma^I_{\alpha\beta}[\psi^\beta, X^I]\right]$$

It is the one-dimensional ${\rm U}(N)$ Yang-Mill theory with maximal supersymmetry.

X^I , ψ^α (for $I = 1, \cdots, 9$ and $\alpha = 1, \cdots, 16$) are $N \times N$ matrices.

$\Gamma^I_{\alpha\beta}$ = 9-dimensional gamma matrices (real symmetric and traceless).

$D_tY = \partial_tY + [A_0, Y]$,   e.o.m of A_0 imposes ${\rm U}(N)$ invariant

$g_{\rm YM}$ dimensionful coupling with energy dimension $3/2$.

Symmetries

The theory has $\mathrm{SO}(9)$ symmetry and 16 supersymmetries.

$$H = \mathrm{Tr}\left(\frac{g_{\mathrm{YM}}^2}{2}\sum_{I=1}^{9}(P^I)^2 - \frac{1}{4g_{\mathrm{YM}}^2}\sum_{I,J=1}^{9}[X^I, X^J]^2 - \frac{i}{g_{\mathrm{YM}}^2}\psi^\alpha \Gamma^I_{\alpha\beta}[\psi^\beta, X^I]\right)$$

$$Q_\alpha = \mathrm{Tr}\left(P^I \Gamma^I_{\alpha\beta}\psi^\beta - \frac{i}{2g_{\mathrm{YM}}^2}[X^I, X^J]\Gamma^{IJ}_{\alpha\beta}\psi^\beta\right), \quad \Gamma^{IJ} = \frac{1}{2}(\Gamma^I\Gamma^J - \Gamma^J\Gamma^I)$$

$$\{Q_\alpha, Q_\beta\} = 2\delta_{\alpha\beta}H\,, \quad [H, Q_\alpha] = 0 \Rightarrow \text{BPS bound } E \geq 0$$

BFSS theory is the unique theory with the above symmetries.

U(1) sector

The U(1) sector is free and decoupled

$$S = \frac{N}{g_{\mathrm{YM}}^2} \int dt \sum_I \left[\frac{(\dot{X}^I)^2}{2} + \psi_\alpha \dot{\psi}_\alpha \right]$$

The fermions ψ_α form a Clifford algebra

$$\{\psi_\alpha, \psi_\beta\} = 2\delta_{\alpha\beta}$$

and give $2^8 = 256$ states.

Ground States

It is believed that the SU(N) part has a single bound state at energy $E = 0$.

Evidence: Witten index $I = 1$ (very subtle due to the flat direction) [Yi, Sethi-Stern, Moore-Nekrasov-Shatashvili, Konechny, Porrati-Rozenberg, Sethi-Stern]

$\Rightarrow$ The ground states of the BFSS matrix quantum mechanics describe a free supersymmetric particle in $\mathbb{R}^9$ with 256 internal states.

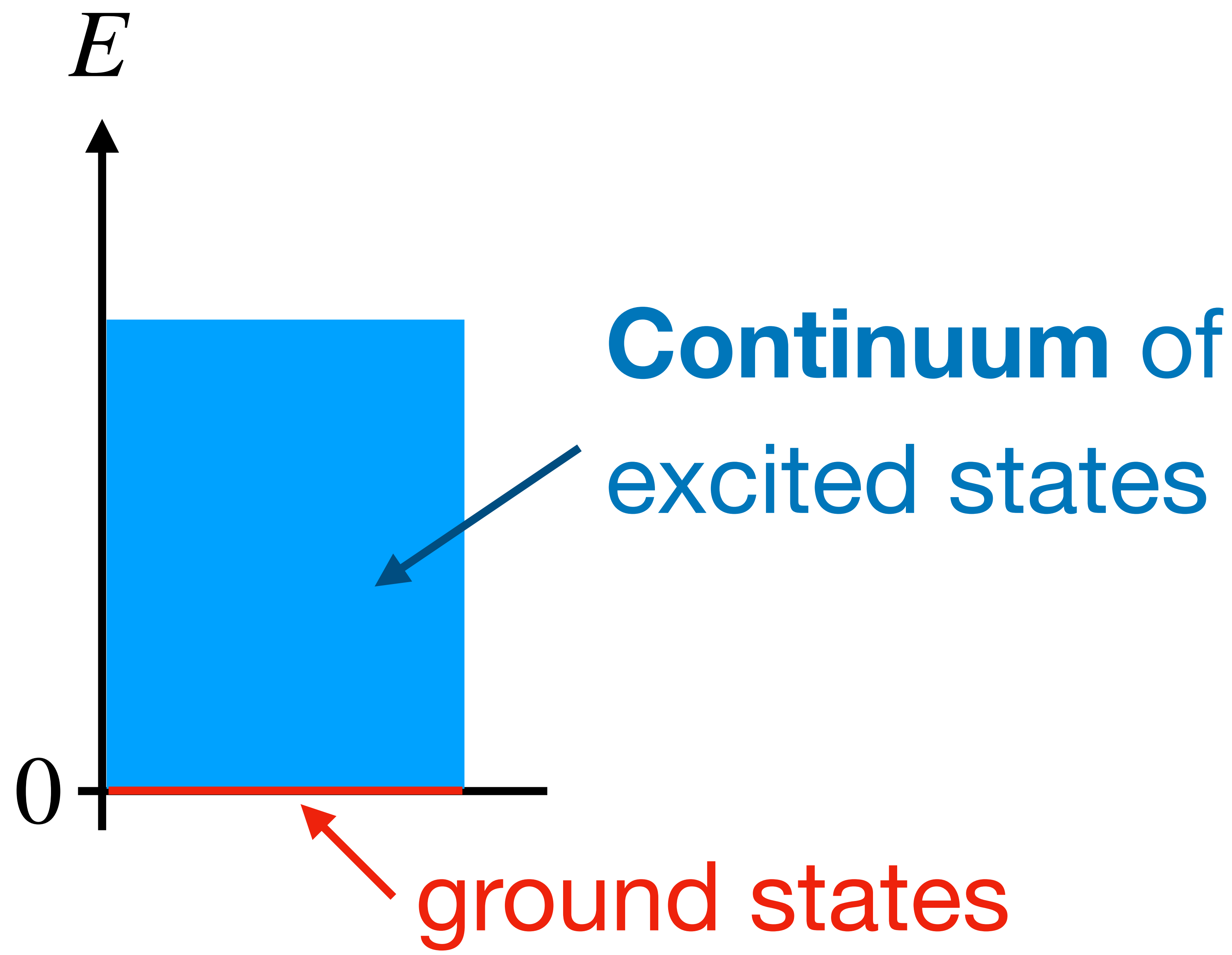
The ground states are the only BPS states (states saturating the BPS bound $E \geq 0$).

Low Energy States

The potential vanishes when the matrices commute: $[X^I, X^J] = 0$

Diagonalize X^I :

$$X^I = \begin{pmatrix} x_1 \mathbf{1}_{N_1} & & & \\ & x_2 \mathbf{1}_{N_2} & & \\ & & x_3 \mathbf{1}_{N_3} & \\ & & & x_4 \mathbf{1}_{N_4} \end{pmatrix}$$



Consider $|x_i - x_j| \gg 1$, and add a bound state wavefunction in each sub-block

$\longrightarrow$ 4 weakly interacting particles

Asymptotic States and Scattering

Asymptotic n particle state: A block diagonal matrix, with n blocks and the center of mass of each block being very far away from the other, and each block of size N_i forms a bound state.

Scattering of the asymptotic state:

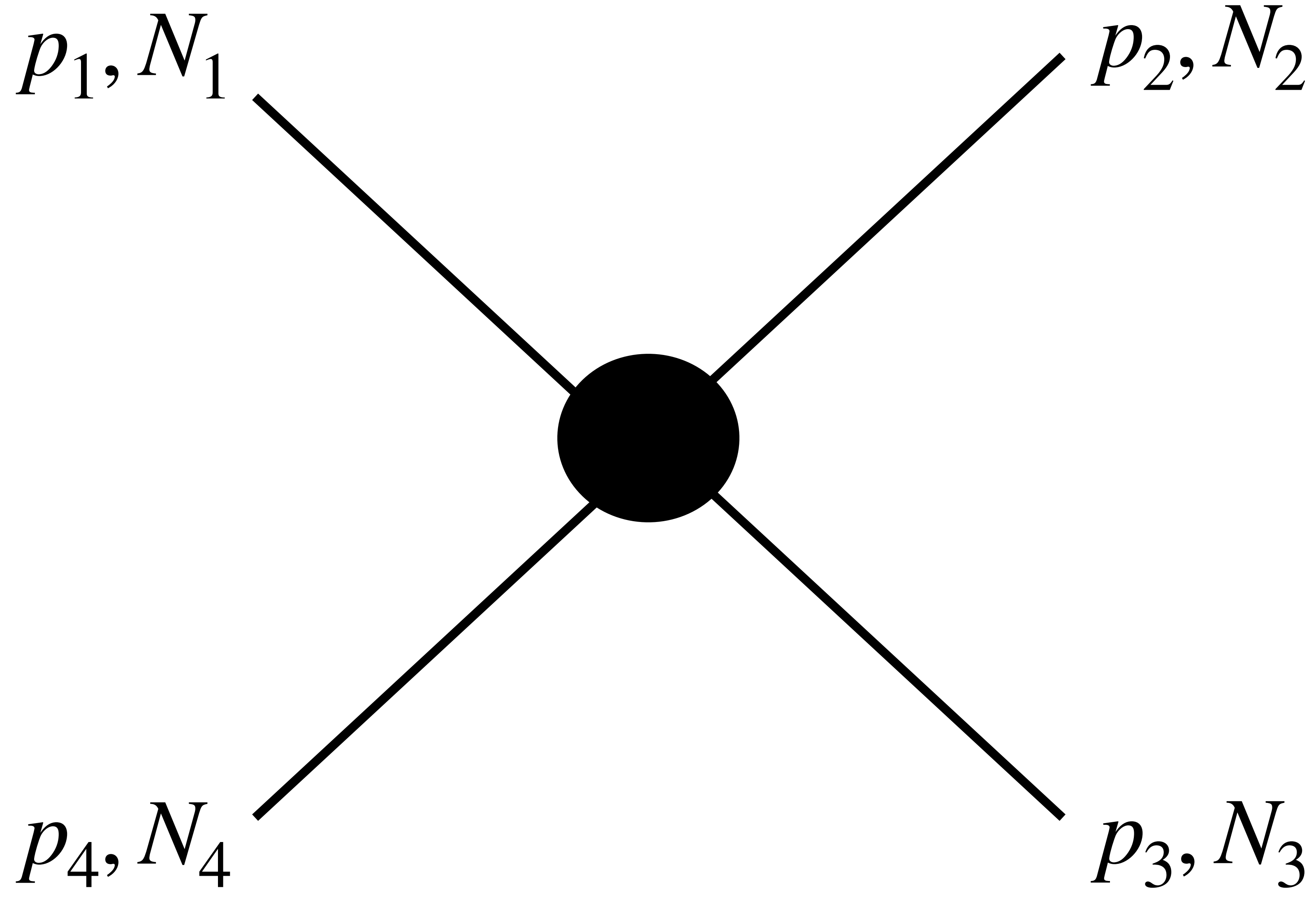
$$\begin{pmatrix} x_1\mathbf{1}_{N_1} & 0 \\ 0 & x_2\mathbf{1}_{N_2} \end{pmatrix}$$

$$\uparrow$$

$$\begin{pmatrix} * & * \\ * & * \end{pmatrix}$$

$$\uparrow$$

$$\begin{pmatrix} x_3\mathbf{1}_{N_3} & 0 \\ 0 & x_4\mathbf{1}_{N_4} \end{pmatrix}$$



BMN Mass Deformation

The BFSS matrix quantum mechanics admits a mass deformation that breaks $\mathrm{SO}(9) \to \mathrm{SO}(3) \times \mathrm{SO}(6)$ but preserves all the 16 supersymmetries.

$$L_{\mathrm{BMN}} = L_{\mathrm{BFSS}} + \frac{1}{g_{\mathrm{YM}}^2}\left[\frac{1}{2}\left(\frac{\mu}{3}\right)^2 \sum_{i=1}^{3} (X^i)^2 + \frac{1}{2}\left(\frac{\mu}{6}\right)^2 \sum_{m=4}^{9} (X^m)^2 + \frac{\mu}{8} i\psi^\alpha \gamma^{123}_{\alpha\beta}\psi^\beta + \frac{\mu}{3} i\epsilon_{ijk}X^iX^jX^k\right]$$

[Berenstein-Maldacena-Nastase]

The mass terms lift all the flat directions, so the spectrum becomes **discrete**.

No asymptotic states. No scattering amplitude. The observables are energy spectrum, correlation functions, …

Supersymmetry

The supersymmetry algebra is deformed by the mass term

$$\{Q_\alpha, Q_\beta\} = 2\delta_{\alpha\beta}H + \mu \times [\mathrm{SO}(3) \times \mathrm{SO}(6) \text{ angular momenta}]$$

$\longrightarrow \mathrm{SU}(2|4)$ group

Pick $Q = Q_1 + iQ_2$ and $Q^\dagger = Q_1 - iQ_2$, we find the BPS bound:

$$\{Q, Q^\dagger\} = 2H - \frac{2\mu}{3}M^{12} - \frac{\mu}{3}M^{45} - \frac{\mu}{3}M^{67} - \frac{\mu}{3}M^{89} \geq 0$$

M^{ij}: angular momentum of the rotation along the ij-plane

Ground States

The bosonic potential is simplified as ($i = 1, 2, 3$ and $m = 4, \cdots, 9$)

$$V_B = \mathrm{Tr}\Big[\frac{1}{2}\left(\frac{\mu}{3}X^i + i\epsilon^{ijk}X^jX^k\right)^2 + \frac{1}{4}(i[X^m, X^n])^2 + \frac{1}{2}(i[X^m, X^i])^2 + \frac{\mu}{72}(X^m)^2\Big]$$

The bosonic potential is minimized at

$$X^m = 0\ , \quad X^i = \frac{\mu}{3}J^i$$ for J^i a representation of $\mathrm{SU}(2)$ algebra.

Ground States

Decompose X^i into $SU(2)$ irreps:

$$X^i = \frac{\mu}{3}\begin{pmatrix} J^i_{N_1} & & \\ & J^i_{N_2} & \\ & & \cdots \end{pmatrix}$$

$$N = n_1 N_1 + n_2 N_1 + n_3 N_3 + \cdots + n_K N_K$$

n_k: number of N_k-dim irreps

Total number of ground states = integer partition $p(N)$: 1, 2, 3, 5, …

Besides the ground states, there are a lot of excited BPS states with $E =$ (angular momenta) > 0. (will see later)

Holography 1:
BFSS Conjecture

IIA and D0 branes

- In type IIA superstring theory (in 10d spacetime $\mathbb{R}^{1,9}$), there are point particles called D0 branes with mass $m_{\mathrm{D0}} = 1/g_{\mathrm{YM}}^2 \ell_s^4$.

- The BFSS matrix quantum mechanics with $N \times N$ matrices describes the **low energy** dynamics of the D0 branes (with energy $E \ll \ell_s^{-1}$).

- n D0 branes can be marginally bound to form a particle of mass nm_{D0}, corresponding to a $n \times n$ block with a bound state wavefunction in BFSS.

M-theory Conjecture

- The 10d type IIA superstring theory is dual to the 11d M-theory compactified on $\mathbb{R}^{1,9} \times \mathrm{S}^1$ with radius $R = m_{\mathrm{D0}}^{-1} = g_{\mathrm{YM}}^2 \ell_s^4$. [Townsend, Witten]
- Small radius $R \to 0$ corresponds to weak string coupling $g_s \sim g_{\mathrm{YM}}^2 \ell_s^3 \to 0$
- Massless excitations of M-theory are 11d supergravitons, whose Kaluza-Klein modes (of the S^1 compactification) have mass

$$\frac{n}{R} = n m_{\mathrm{D0}} = \text{mass of n D0 bround bound state}$$

Conjecture: KK supergravitons $\longleftrightarrow$ D0 branes

- 11d supergravity multiplet contains 256 states that matches with the 256 ground states in the BFSS theory.

Null Compactification

Consider M-theory compactified on a tiny circle $R_s \to 0$ and focus on a sector with $p^1 = \frac{N}{R_s}$, $H = p^0 - \frac{N}{R_s} \equiv h\frac{R_s}{\ell_P^2}$ with N, ℓ_P, h fixed, corresponding to a N D0 brane bound state with excitation $E \sim R_s/\ell_P^2 \ll R_s^{\frac{1}{2}}/\ell_P^{\frac{3}{2}} = \ell_s^{-1}$, so it is described by the BFSS theory. Now, consider a large boost

$$\begin{pmatrix} x \\ t \end{pmatrix} \sim \begin{pmatrix} x \\ t \end{pmatrix} + \begin{pmatrix} R_s \\ 0 \end{pmatrix} \xrightarrow[R_s \to 0]{\text{a large boost} \; \beta = \frac{v}{c} = R/\sqrt{R^2 + 2R_s^2}} \begin{pmatrix} x \\ t \end{pmatrix} \sim \begin{pmatrix} x \\ t \end{pmatrix} + \begin{pmatrix} R/\sqrt{2} + R_s^2/\sqrt{2}R \\ -R/\sqrt{2} \end{pmatrix}$$

$\Rightarrow$ null compactified M-theory with $p^- \approx -h\frac{R}{\ell_P^2}$, $p^+ \approx \frac{N}{R}$ [Seiberg]

$$p^\pm = (p^1 \pm p^0)/\sqrt{2}$$

BFSS conjecture

BFSS matrix quantum mechanics (with Hamiltonian H) is dual to a fixed null momentum p^+ sector in M-theory compactified on a null circle with radius R.

$$\frac{1}{g_{\mathrm{YM}}^2} = \frac{\ell_P^6}{R^3}, \quad p^+ = \frac{N}{R}, \quad p^- = -H$$

[Susskind, Sen, Seiberg]

The null circle can be decompactified by the limit

$$N, R \to \infty \text{ with } p^+ \text{ fixed}$$

[Banks-Fischler-Shenker-Susskind]

Some Recent Evidence for the Conjecture

- Recently, the M-theory three-graviton amplitude was shown to exactly match with the corresponding amplitude in the BFSS matrix quantum mechanics. [Maldacena-Herderschee]

- In M-theory on $\mathbb{R}^{1,10}$, this amplitude is completely fixed by supersymmetry and the SO(1,10) Lorentz symmetry.

- However, only the SO(9) is manifest in BFSS, so the computation is nontrivial.

- This result was used to argue that the higher-point amplitudes in BFSS are Lorentz symmetric. [Maldacena-Herderschee 2]

Plane-wave Background

The mass deformation from BFSS to BMN corresponds to deforming the flat space to the plane-wave background [Berenstein-Maldacena-Nastase]

$$ds^2 = -2dtdx^- + \sum_{I=1}^{9}(dx^I)^2 - \left[\left(\frac{\mu}{3}\right)^2 \sum_{i=1}^{3}(x^i)^2 + \left(\frac{\mu}{6}\right)^2 \sum_{m=4}^{9}(x^m)^2\right] dt^2$$

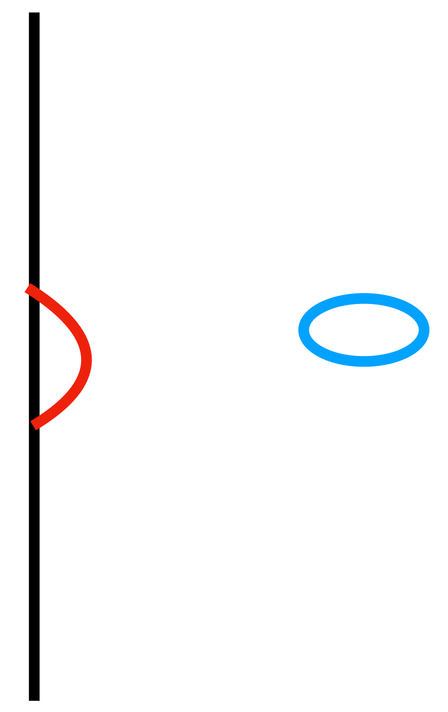
There is a potential wall at $|x^i|, |x^m| \to \infty$. This agrees with the potential in the BMN matrix quantum mechanics.

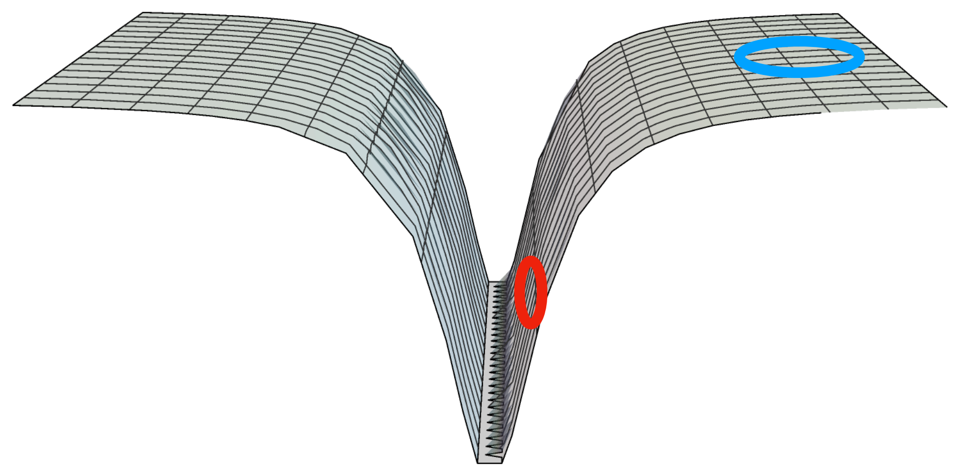
Holography 2:
Gauge/Gravity Duality

D-brane as Solitons

- D-branes are dynamical objects and can curve the spacetime. When the number N of the D-branes becomes large, the system admits an effective description as a black brane solution in supergravity.

- The open string excitations on the D-brane become closed string near the horizon.





Low Energy/Near Horizon Limit

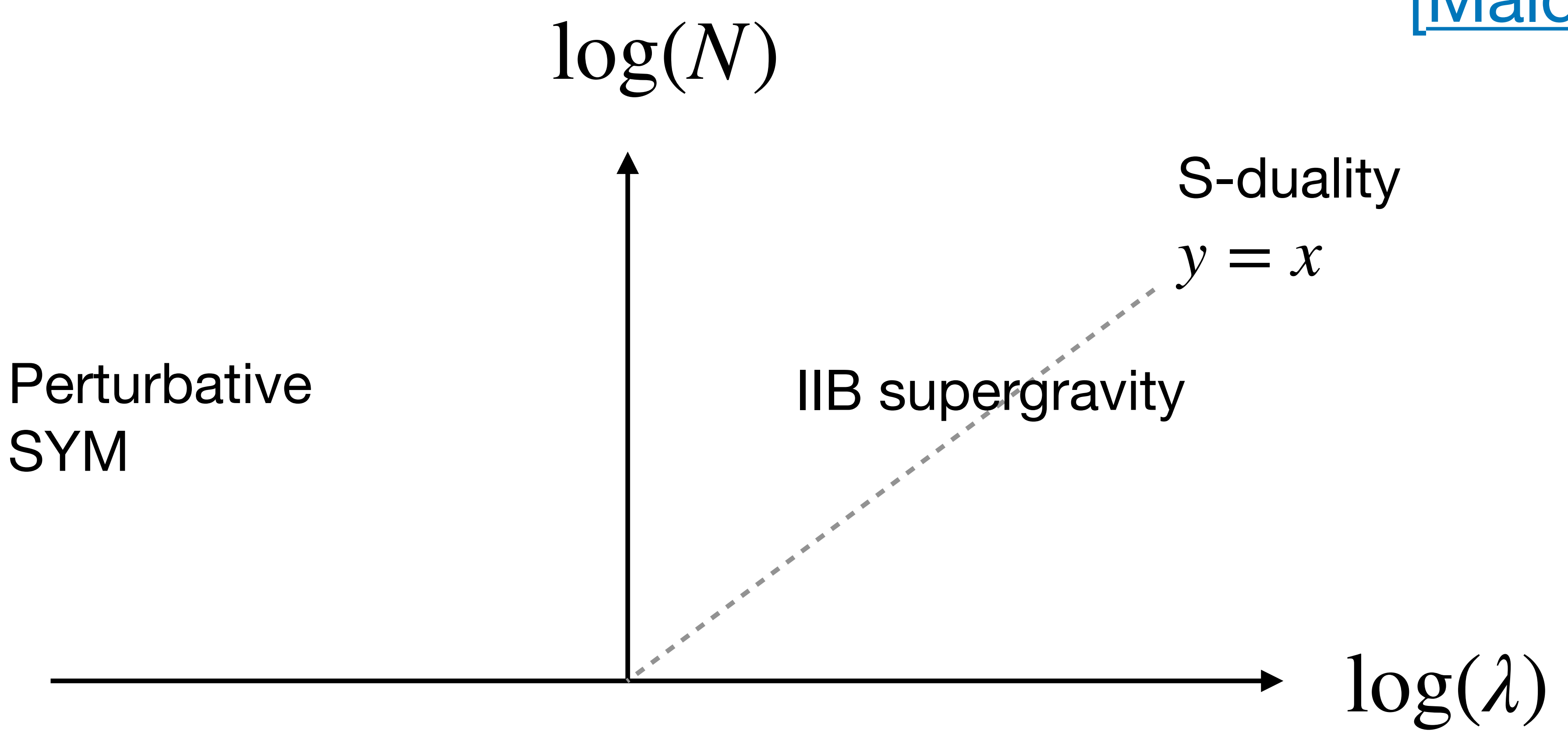
- Low energy limit on the D-brane side: the closed strings ⬭ decouple. The theory is described by the SYM.

- The equivalent near horizon limit on the brack brane side: the far closed strings ⬭ decouple. The theory is described by a closed string theory on the near horizon geometry.

- Applying this procedure to the D3 brane system $\longrightarrow$ duality between $\mathcal{N} = 4$ SYM and IIB string on $\mathrm{AdS}_5 \times S^5$

[Maldacena]

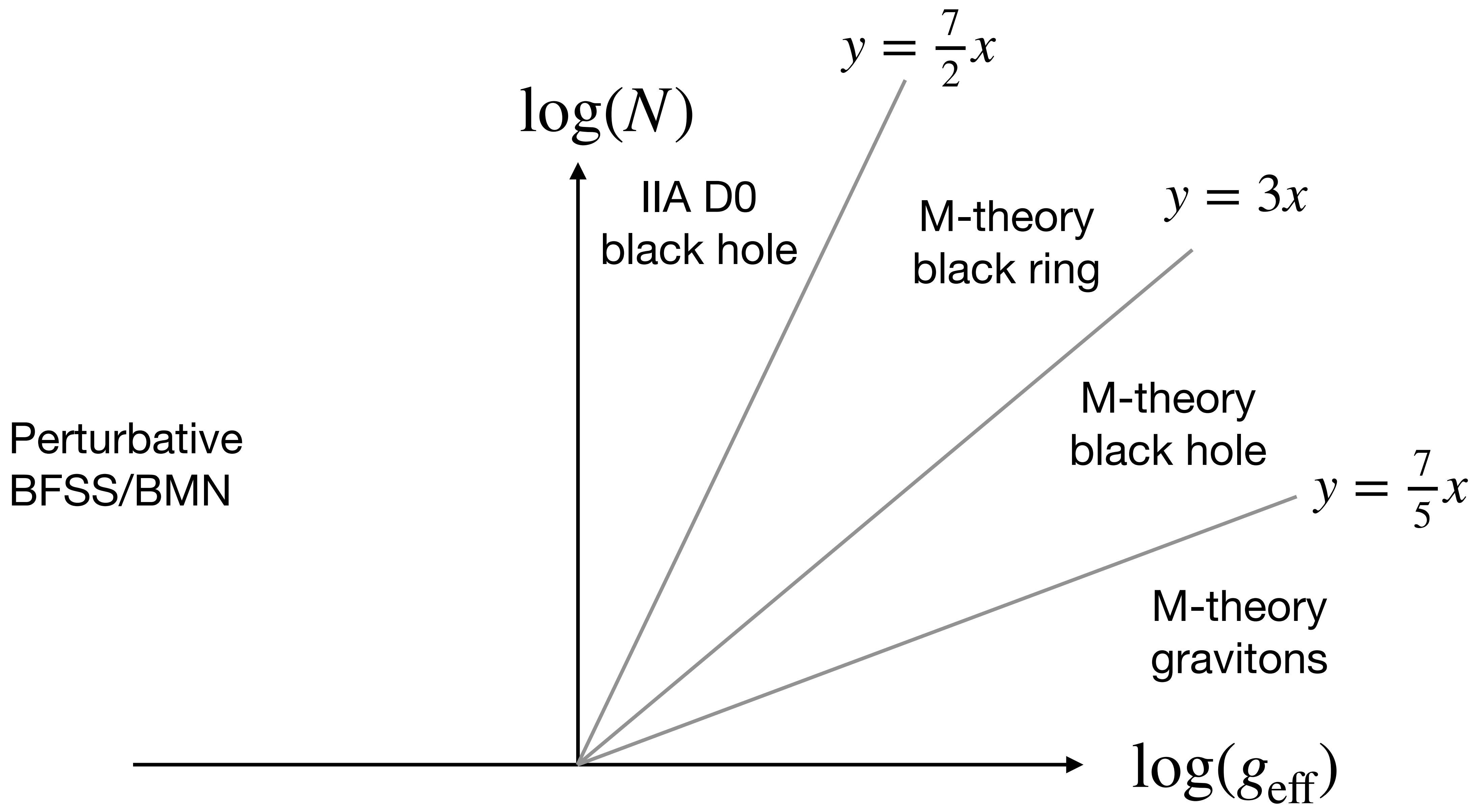
$$\lambda = g_{\mathrm{YM}}^2 N = \left(\frac{\ell_{\mathrm{AdS}}}{\ell_s}\right)^4$$



Gauge/Gravity Duality for D0

- Unlike 4d (D3 case), the 1d Yang-Mills coupling is dimensionful. At an energy scale E (can choose $E = \mu$ for BMN), the (effective) coupling is

$$g_{\text{eff}}^2 = g_{\text{YM}}^2 N E^{-3} \propto R^3, \quad R\text{: radius of the M-theory } S^1$$



[Itzhaki-Maldacena-Sonnenschein-Yankielowycz]

Black Holes in Matrix QM

- The D0 black hole corresponds to a matrix with a large $N \times N$ black. The black hole is asymptotically flat, so it is unstable and can decay by emitting D0 branes, corresponding to the splitting process

$$(X^I)_{N\times N} \longrightarrow \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & (X^I)_{(N-2)\times(N-2)} \end{pmatrix}$$

- Turning on the mass μ corresponds to turning on a potential wall at infinity, so the black hole becomes stable. Counting its microstates becomes a well-defined problem.

Counting black hole microstates via BMN

Counting States

- We can reliably count the states in the BMN matrix quantum mechanics at weak coupling $g_{\rm eff} \ll 1$.

- However, the black holes live at strong coupling.

- Resolution: A particular way of counting states is independent of $g_{\rm eff}$.

Witten index = (# of bosonic BPS states) - (# of fermionic BPS states)

Witten Index

- Recall the BPS bound ($Q = Q_{\alpha=1} + iQ_{\alpha=2}$):

$$2\Delta \equiv \{Q, Q^\dagger\} = 2H - \frac{2\mu}{3}M^{12} - \frac{\mu}{3}M^{45} - \frac{\mu}{3}M^{67} - \frac{\mu}{3}M^{89} \geq 0$$

- The partition function (temperature $T = 1/\beta$):

$$Z = \mathrm{Tr}\,\Omega\,, \quad \Omega \equiv e^{-\beta\Delta - 2\omega M^{12} - \Delta_1 M^{45} - \Delta_2 M^{67} - \Delta_3 M^{89}}$$

- Let us choose the parameters ω, Δ_i such that $\{Q, \Omega\} = 0$

$$\Delta_1 + \Delta_2 + \Delta_3 - 2\omega = 2\pi i$$

Witten Index

- The Witten index is a specialization of the partition function

$$I = Z\Big|_{\{Q,\Omega\}=0} = \mathrm{Tr}\left[\underbrace{(-1)^F}_{=e^{2\pi i M^{12}}} e^{-\beta\Delta-\Delta_1(M^{12}+M^{45})-\Delta_2(M^{12}+M^{67})-\Delta_3(M^{12}+M^{89})}\right]$$

- Bosonic and fermionic states contribute with opposite signs. States with $\Delta > 0$ form doublets $|\Psi\rangle, Q|\Psi\rangle$ with canceling contribution.

- Witten index only counts BPS states ($\Delta = 0$) and is independent of β.

- The angular momenta M^{ij} are independent of g_{eff}, so as the Witten index.

Computation at Weak Coupling

- We consider weak coupling $g_{\text{eff}} \to 0$ ($\mu \to \infty$). The system reduces to many harmonic oscillators, and the spectrum was worked out in [Dasgupta, Sheikh-Jabbari, Raamsdonk].

- Using this result, the Witten index can be computed straightforwardly.

- At $\mu \to \infty$, tunneling between different vacua requires infinite energy. The system splits into superselection sectors. The Witten index is the sum

$$I = \sum_{n_i, N_i} I_{n_i; N_i} \qquad \left(N = \sum_{i=1}^{K} n_i N_i \,, \quad n_i \text{ number of } N_i\text{-dim SU(2) irreps}\right)$$

Matrix sphrical harmonics $Y^{N_k N_l}_{j,m}$: $N_k \times N_l$ matrix as a spin-j irrep in the tensor product of spin-$\left(\frac{N_k - 1}{2}\right)$ and spin-$\left(\frac{N_l - 1}{2}\right)$ irreps.

Decompose the (k, l)-th block: $X^a_{kl} = \Sigma_{j,m} (x^a_{kl})_{jm} \otimes Y^{N_k N_l}_{jm}$ and similar for X^i and ψ_α . (Note that one set of modes in X^i are pure gauges.)

Creation operators in the $\{n_i, N_i\}$ sector: [Dasgupta, Sheikh-Jabbari, Raamsdonk]

Type	Label	Mass	Spins	$SO(6) \times SO(3)$
$S0(6)$	$(x^a_{kl})_{jm}$	$\frac{1}{6} + \frac{j}{3}$	$\frac{1}{2}\lvert N_k - N_l\rvert \le j \le \frac{1}{2}(N_k + N_l) - 1$	$(6, 2j+1)$
$S0(3)$	α^{jm}_{kl}	$\frac{1}{3} + \frac{j}{3}$	$\frac{1}{2}\lvert N_k - N_l\rvert - 1 \le j \le \frac{1}{2}(N_k + N_l) - 2$	$(1, 2j+1)$
	β^{jm}_{kl}	$\frac{j}{3}$	$\frac{1}{2}\lvert N_k - N_l\rvert + 1 \le j \le \frac{1}{2}(N_k + N_l)$	$(1, 2j+1)$
Fermions	$\chi^{I\,jm}_{kl}$	$\frac{1}{4} + \frac{j}{3}$	$\frac{1}{2}\lvert N_k - N_l\rvert - \frac{1}{2} \le j \le \frac{1}{2}(N_k + N_l) - \frac{3}{2}$	$(\bar{4}, 2j+1)$
	$\eta^{jm}_{I\,kl}$	$\frac{1}{12} + \frac{j}{3}$	$\frac{1}{2}\lvert N_k - N_l\rvert + \frac{1}{2} \le j \le \frac{1}{2}(N_k + N_l) - \frac{1}{2}$	$(4, 2j+1)$

- Witten index: [CMC]

index of a single creation operator

$$I_{n_i;N_i} = \int \prod_{k=1}^{K} [dU_k] \exp \left[\sum_{m=1}^{\infty} \sum_{k,l=1}^{K} \frac{1}{m} \iota_{kl}(m\Delta_i) \underbrace{\mathrm{Tr}\, U_k^{\dagger m} \mathrm{Tr}\, U_l^m} \right]$$

$\mathrm{U}(n_k) \times \mathrm{U}(n_l)$ bifundamental character

$$\iota_{kl}(\Delta_i) = \sum_{j=\frac{1}{2}|N_k-N_l|}^{\frac{1}{2}(N_k+N_l)-1} (-1)^{2j+1} e^{-j(\Delta_1+\Delta_2+\Delta_3)} (1-e^{-\Delta_1})(1-e^{-\Delta_2})(1-e^{-\Delta_3}) + \delta_{N_k,N_l}$$

- The integral of $n_k \times n_k$ unitary matrix U_k impose the gauge invariance under the $\mathrm{U}(n_1) \times \cdots \times \mathrm{U}(n_K)$ remaining gauge symmetry of the $\{n_i, N_i\}$ sector.

Trivial Vacuum Sector

- Let us focus on the sector of the trivial vacuum ($X^i = 0$ trivial SU(2) rep).

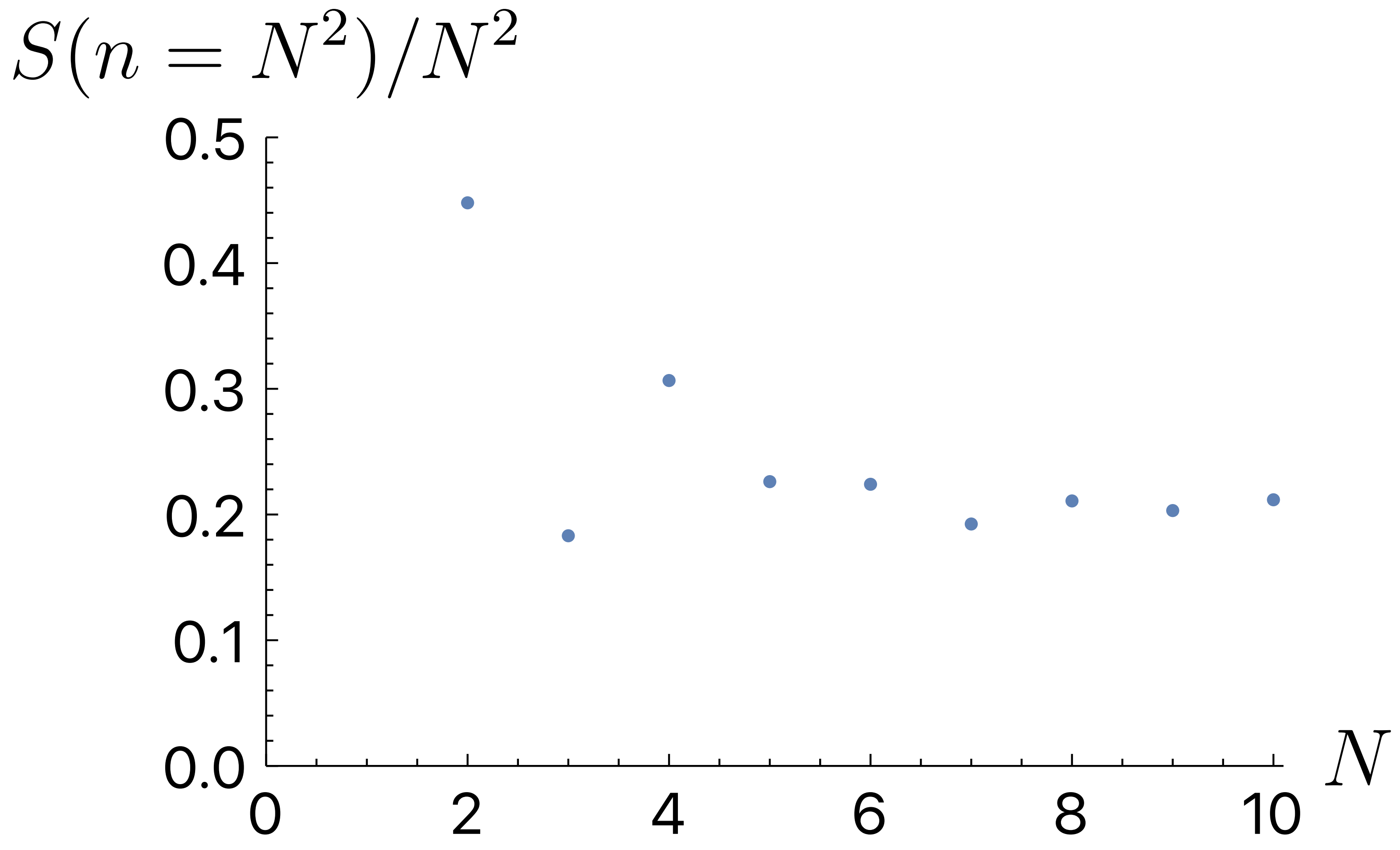
$$I_{N;1} = \int [dU] \exp \left\{ \sum_{m=1}^{\infty} \frac{1-(1-e^{-m\Delta_1})(1-e^{-m\Delta_2})(1-e^{-m\Delta_3})}{m} \mathrm{Tr}\, U^{\dagger m} \mathrm{Tr}\, U^m \right\}$$

- Let us specialize and expand the formula as

$$I_{N;1} = \sum_n d_n t^n\,, \quad t^2 = e^{-\Delta_1} = e^{-\Delta_2} = e^{-\Delta_3}, \quad n = \text{angular momenta}$$

- The coefficients d_n can be computed by explicitly evaluating the integral.

- The entropy is given by $S(n) = \log(|d_n|)$. We find numerically that the entropy growth as $S \sim N^2$ at $n \sim N^2$.



- This agrees with the entropy of the expected black hole in 10d supergravity from the Bekenstein-Hawking formula $S = A/4G_N$.

- The black hole should preserve 2 supersymmetries. However, no such black hole is known so far.

Conclusions

- The BFSS theory is an important window into flat space non-perturbative physics.
- The BMN theory is a simple model for black hole microstates.
- Open problems:
  - BPS black hole soluiton
  - State counting in other sectors
  - Evaluation of the index at large N
  - Witten index of D0-D4 (8 SUSY) matrix quantum mechanics with mass deformation. Relation to instanton partition function?

Thank you